Discrete functional analysis tools for Discontinuous Galerkin methods with application to the incompressible Navier–Stokes equations

Daniele Antonio Di Pietro, Alexandre Ern

DISCRETE FUNCTIONAL ANALYSIS TOOLS FOR DISCONTINUOUS GALERKIN METHODS WITH APPLICATION TO THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

DANIELE A. DI PIETRO[1] AND ALEXANDRE ERN[2]

Abstract. Two discrete functional analysis tools are established for spaces of piecewise polynomial functions on general meshes: (i) a discrete counterpart of the continuous Sobolev embeddings, in both Hilbertian and non-Hilbertian settings; (ii) a compactness result for bounded sequences in a suitable Discontinuous Galerkin norm, together with a weak convergence property for some discrete gradients. The proofs rely on techniques inspired by the Finite Volume literature, which differ from those commonly used in Finite Element analysis. The discrete functional analysis tools are used to prove the convergence of Discontinuous Galerkin approximations of the steady incompressible Navier–Stokes equations. Two discrete convective trilinear forms are proposed, a non-conservative one relying on Temam's device to control the kinetic energy balance and a conservative one based on a nonstandard modification of the pressure.

1. Introduction

Discontinuous Galerkin (DG) methods were introduced over thirty years ago to approximate hyperbolic and elliptic PDE's (see e.g. [2, 16] for a historical perspective), and they have received extensive attention over the last decade. For linear PDE's, the mathematical analysis of such methods is well–understood; see e.g. [2] for a unified analysis for the Poisson problem, [14] for advection–diffusion equations with semidefinite diffusion, and [16, 17, 18] for a unified analysis encompassing hyperbolic and elliptic PDE's in the framework of Friedrichs' systems. The situation is substantially different when dealing with *nonlinear* second-order PDE's. Indeed, although DG methods have been widely used for such problems, their mathematical analysis has hinged almost exclusively on strong regularity assumptions on the exact solution. This is in stark contrast with the recent literature on Finite Volume (FV) schemes where, following the penetrating works of Eymard, Gallouët, Herbin and co-authors (see e.g. [20, 21, 22]), new discrete functional analysis tools have been derived allowing to prove the convergence to minimum regularity solutions, i.e. solutions belonging to the natural function spaces in which the weak formulation of the PDE is set. The key ideas can be summarized as follows:

(i) an a priori estimate on the discrete solution and an associated compactness result are used to infer the strong convergence of a subsequence of discrete solutions to a function u in some Lebesgue space, say $L^2(\Omega)$;

(ii) the construction of a discrete gradient converging to ∇u in a suitable Lebesgue space allows to prove that the limit u actually belongs to some space with additional regularity, say $H^1_0(\Omega)$;

Date: Draft version: May 14, 2008.

(iii) the convergence of the scheme is finally proved testing against a discrete projection of a smooth function belonging to some convenient dense subspace, say $C_c^\infty(\Omega)$.

When the exact solution is unique, the convergence of the whole sequence of discrete approximations is deduced. Moreover, stronger convergence results on the discrete gradient can be derived using the dissipative structure of the problem at hand whenever available. In the present work we show how the analysis tools derived for FV schemes can be extended to DG methods using the steady incompressible Navier-Stokes equations as a model problem. Discontinuous Galerkin approximations of the steady incompressible Navier–Stokes problem have been derived in recent works using different techniques; see, among others, [3, 9, 25, 26, 30].

The present analysis relies on two discrete functional analysis tools in piecewise polynomial spaces on general meshes of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ (DG spaces henceforth). Firstly, upon introducing the usual $\|\cdot\|_{\mathrm{DG}}$-norm consisting of the broken gradient plus a jump term (see (5)) as well as non-Hilbertian variants thereof (see §6), we prove discrete Sobolev embeddings that are the counterpart of those valid at the continuous level,

$$\|v\|_{L^q(\Omega)} \leq \mathcal{S}_{p,q}\|\nabla v\|_{L^p(\Omega)^d}, \quad \forall v \in W_0^{1,p}(\Omega),$$

for suitable indices q and p. Probably the best known discrete embedding of such a type is the so-called broken Poincaré–Friedrichs inequality obtained with $p = q = 2$; see e.g. [1, 5]. The broken Sobolev embeddings we are concerned with have been derived by Lasis and Süli [28] in the Hilbertian case ($p = 2$). Those we establish in a non-Hilbertian setting ($p \neq 2$) are, to the best of our knowledge, new. The proofs are substantially different from the ones used in the finite element literature, which rely on elliptic regularity and or on nonconforming finite element interpolants. Indeed, we take inspiration from the techniques used in [21] in the case of piecewise constant functions. A crucial point is the observation that the BV norm defined in Lemma 6.2 hereafter is controlled by the $\|\cdot\|_{\mathrm{DG}}$-norm and also by non-Hilbertian variants thereof. The present technique of proof readily incorporates the use of general meshes, an important feature when working with DG methods. Observe, however, that we only establish the embedding results in DG spaces only, and not in the larger setting of broken Sobolev spaces. The latter are indeed not used in the convergence proofs below.

The second functional analysis tool, which is, to the best of our knowledge, new in the framework of DG methods, is a compactness result for bounded sequences in the $\|\cdot\|_{\mathrm{DG}}$-norm and non-Hilbertian versions thereof. Here again, the proof is quite simple and it is inspired from [21]: it consists of using Kolmogorov's Compactness Criterion (see e.g. [7, Theorem IV.25]) based on uniform translates estimates in $L^1(\mathbb{R}^d)$ together with the above discrete Sobolev embeddings and a discrete gradient operator that is shown to be weakly convergent in some $L^p(\Omega)$ space with $p > 1$.

This paper is organized as follows. §2 introduces the discrete setting, including the assumptions on the meshes, the DG spaces, and the discrete gradient operators, whose weak convergence is proven in Theorem 2.2. §3 is concerned with the Poisson problem; its purpose is to show how the diffusive term is analyzed. The main result is Theorem 3.1. §4 deals with the Stokes equations; its purpose is to show how the velocity–pressure coupling is handled. The main result is Theorem 4.1. §5 is concerned with the steady incompressible Navier–Stokes equations;

its main result is Theorem 5.1. Two discrete convective trilinear forms are proposed, a non-conservative one relying on Temam's device to control the kinetic energy balance [32] and a conservative one based on a nonstandard modification of the pressure hinted to in [9]. Finally, §6 contains the discrete functional analysis tools in DG spaces. The main results are Theorem 6.1 and 6.3 which are presented in a non-Hilbertian setting since their validity extends beyond the model problems considered in this work.

2. The discrete setting

2.1. **Meshes.** Let Ω be an open bounded connected subset of $\mathbb{R}^d$ ($d > 1$) whose boundary is a finite union of parts of hyperplanes.

Definition 2.1 (Admissible meshes)**.** *Let $\mathcal{H}$ be a countable set. The family $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ is said to be an* admissible *mesh family if the following assumptions are satisfied:*

(i) *for all $h \in \mathcal{H}$, $\mathcal{T}_h$ is a finite family of non-empty connex (possibly non-convex) open disjoint sets T forming a partition of Ω and whose boundaries are a finite union of parts of hyperplanes;*
(ii) *there is a parameter N_∂, independent of h, such that each $T \in \mathcal{T}_h$ has at most N_∂ faces. A set $F \subset \partial T$ is said to be a face of T is F is part of a hyperplane, and if either F is located on the boundary of Ω or there is $T' \in \mathcal{T}_h$ such that $F = \partial T \cap \partial T'$;*
(iii) *there is a parameter ϱ_1 independent of h such that for all $T \in \mathcal{T}_h$,*

$$\sum_{F\subset\partial T} h_F|F| \leq \varrho_1|T|, \tag{1}$$

where h_F denotes the diameter of the face F, $|F|$ its $(d-1)$-dimensional measure and $|T|$ the d-dimensional measure of T;
(iv) *for all $h \in \mathcal{H}$, each $T \in \mathcal{T}_h$ is affine-equivalent to an element belonging to a finite collection of reference elements;*
(v) *the ratio of the diameter h_T of any $T \in \mathcal{T}_h$ to the diameter of the largest ball inscribed in T is bounded from above by a parameter ϱ_2 independent of h;*
(vi) *there is a parameter ϱ_3, independent of h, such that for all $T \in \mathcal{T}_h$ and for all faces $F \subset \partial T$, $h_F|F| \geq \varrho_3|T|$.*

For each $h \in \mathcal{H}$, we define $\mathrm{size}(\mathcal{T}_h) \stackrel{\mathrm{def}}{=} \max_{T\in\mathcal{T}_h} h_T$. *The parameters introduced in the above definition will be referred to as the basic mesh parameters and collectively denoted by the symbol $\mathcal{P}$.*

Remark 2.1. Assumption (vi) is needed only when working with a particular choice of the stabilization bilinear form penalizing interelement jumps. It can be lifted by working with other forms; see Remark 3.2 for further discussion. Furthermore, assumption (v) will not be needed in §6 to prove the discrete Sobolev embeddings nor the weak convergence of discrete gradients.

Figure 1 presents an example of admissible mesh in two space dimensions. The mesh faces are collected in the set $\mathcal{F}_h$. The set $\mathcal{F}_h$ will be partitioned into $\mathcal{F}_h^i \cup \mathcal{F}_h^b$, where $\mathcal{F}_h^b$ collects the faces located on the boundary of Ω and $\mathcal{F}_h^i$ the remaining ones. For $F \in \mathcal{F}_h^i$, there are T_1 and T_2 in $\mathcal{T}_h$ such that $F = \partial T_1 \cap \partial T_2$, and we define ν_F as the unit normal vector to F pointing from T_1 to T_2. For any function

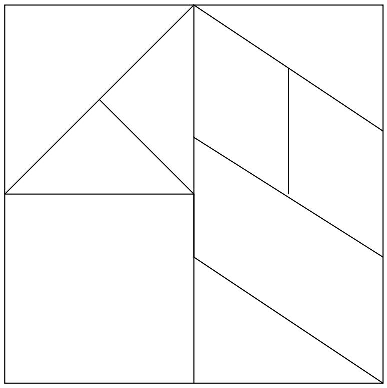


FIGURE 1. An example of admissible mesh

φ such that a (possibly two-valued) trace is defined on F, let

$$[\![\varphi]\!] \stackrel{\text{def}}{=} \varphi_{|T_1} - \varphi_{|T_2}, \qquad \{\!\{\varphi\}\!\} \stackrel{\text{def}}{=} \frac{1}{2}(\varphi_{|T_1} + \varphi_{|T_2}). \tag{2}$$

For $F \in \mathcal{F}_h^b$, ν_F is defined as the unit outward normal to Ω, while the jump and average are conventionally defined as $[\![\varphi]\!] \stackrel{\text{def}}{=} \varphi$ and $\{\!\{\varphi\}\!\} \stackrel{\text{def}}{=} \varphi$.

For any integer $k \geq 0$ and for all $T \in \mathcal{T}_h$, let $\mathbb{P}_k(T)$ denote the vector space of polynomial functions defined on T with real coefficients and with total degree less than or equal to k. Owing to assumptions (iii) and (iv) in Definition 2.1, there is $c_{k,\mathcal{P}}$ such that, for all $h \in \mathcal{H}$ and for all $T \in \mathcal{T}_h$,

$$\forall v_h \in \mathbb{P}_k(T), \qquad \sum_{F \subset \partial T} h_F \int_F |v_h|^2 \leq c_{k,\mathcal{P}} \int_T |v_h|^2. \tag{3}$$

Here and in what follows, the symbol c will be used to denote a positive generic constant whose value can change at each occurrence. To keep track of the dependency of such constants on some parameters, subscripts will be used whenever relevant.

2.2. **DG spaces.** Let $k \geq 0$ and consider the finite dimensional space

$$V_h^k \stackrel{\text{def}}{=} \{v_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ v_{h|T} \in \mathbb{P}_k(T)\}. \tag{4}$$

For $k \geq 1$, this space is equipped with the norm

$$\|v_h\|_{\mathrm{DG}}^2 \stackrel{\text{def}}{=} \sum_{T \in \mathcal{T}_h} \int_T |\nabla v_h|^2 + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \int_F |[\![v_h]\!]|^2, \tag{5}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. For further use, it will be convenient to introduce the seminorm

$$|v_h|_{\mathrm{J},\mathcal{F},\pm 1}^2 \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}} h_F^{\pm 1} \int_F |[\![v_h]\!]|^2, \tag{6}$$

where $\mathcal{F}$ is a subset of $\mathcal{F}_h$ that will usually be taken equal to $\mathcal{F}_h$ or to $\mathcal{F}_h^i$. Moreover, we define $\nabla_h v_h$ as the piecewise gradient of $v_h \in V_h^k$, i.e., $\nabla_h v_h \in [V_h^{k-1}]^d$ is such that for all $T \in \mathcal{T}_h$, $\nabla_h v_{h|T} = \nabla(v_{h|T})$, so that

$$\|v_h\|_{\mathrm{DG}}^2 = \|\nabla_h v_h\|_{L^2(\Omega)^d}^2 + |v_h|_{\mathrm{J},\mathcal{F}_h,-1}^2. \tag{7}$$

The above norm and seminorm can be extended to $H^1_+(\Omega) \stackrel{\text{def}}{=} H^1_0(\Omega)+V_h^k$ (actually, an extension to $C_c^\infty(\Omega)+V_h^k$ is sufficient for the present purposes).

A straightforward but important result concerns the approximability of smooth functions in the $\|\cdot\|_{\mathrm{DG}}$-norm. For all $l \geq 0$, let π_h^l denote the $L^2(\Omega)$-orthogonal projection from $L^2(\Omega)$ onto V_h^l. Let $\varphi \in C_c^\infty(\Omega)$. Then, owing to assumptions (iii)–(v) in Definition 2.1, it is clear using classical approximation properties (see e.g. [6, 15]) that for all $l \geq 1$,

$$\|\varphi - \pi_h^l \varphi\|_{\mathrm{DG}} \to 0 \quad \text{as size}(\mathcal{T}_h) \to 0. \tag{8}$$

In what follows, we shall make frequent use of the projector π_h^1 which will be simply denoted by π_h. For $l=0$ we shall use the following property:

$$\|\varphi - \pi_h^0 \varphi\|_{L^\infty(\Omega)} \to 0 \quad \text{as size}(\mathcal{T}_h) \to 0. \tag{9}$$

The above projectors will also be applied componentwise to vector-valued functions. The following stability result holds: For all $v \in H^1(\Omega)$,

$$\|\nabla_h \pi_h^k v\|^2_{L^2(\Omega)^d} + \sum_{F\in\mathcal{F}_h} h_F^{-1} \|v - \pi_h^k v\|^2_{L^2(F)} \leq c_{k,\mathcal{P}} \|v\|^2_{H^1(\Omega)}. \tag{10}$$

For ease of exposition, we restate hereafter the consequences of Theorem 6.1 in the Hilbertian setting. The proof will be given in §6.

Theorem 2.1 (Discrete Sobolev embeddings)**.** *For all q such that*

(i) $1 \leq q \leq \frac{2d}{d-2}$ *if* $d \geq 3$*;*
(ii) $1 \leq q < +\infty$ *if* $d = 2$*;*

there is σ_q such that

$$\forall v_h \in V_h^k, \qquad \|v_h\|_{L^q(\Omega)} \leq \sigma_q \|v_h\|_{\mathrm{DG}}. \tag{11}$$

The constant σ_q additionally depends on k, $|\Omega|$, and $\mathcal{P}$.

2.3. **Discrete gradient operators.** For all $F \in \mathcal{F}_h$, let $r_F^l : L^2(F) \to [V_h^l]^d$, $l \geq 0$, be the lifting operator defined as follows: For all $\phi \in L^2(F)$,

$$\forall \tau_h \in [V_h^l]^d, \qquad \int_\Omega r_F^l(\phi)\cdot\tau_h = \int_F \{\!\{\tau_h\}\!\}\cdot\nu_F \phi. \tag{12}$$

Clearly, the support of $r_F^l(\phi)$ consists of the one or two mesh elements of which F is a face. For $v \in H^1_+(\Omega)$, define

$$R_h^l(v) \stackrel{\text{def}}{=} \sum_{F\in\mathcal{F}_h} r_F^l(v). \tag{13}$$

Let now $k \geq 1$. The following discrete gradient operators $G_h^l : V_h^k \to [V_h^{\max(k-1,l)}]^d$ will play an important role in the subsequent analysis

$$\forall v_h \in V_h^k, \qquad G_h^l(v_h) \stackrel{\text{def}}{=} \nabla_h v_h - R_h^l([\![v_h]\!]) = \nabla_h v_h - \sum_{F\in\mathcal{F}_h} r_F^l([\![v_h]\!]). \tag{14}$$

For a given $k \geq 1$, the most natural value for l is k or $(k-1)$, but the values $l=0$ and $l=2k$ will also be used.

Proposition 2.1 (Stability of discrete gradients)**.** *Let $k \geq 1$ and let $l \geq 0$. Then,*

$$\forall v_h \in V_h^k, \qquad \|G_h^l(v_h)\|_{L^2(\Omega)^d} \leq c_{k,l,\mathcal{P}} \|v_h\|_{\mathrm{DG}}. \tag{15}$$

Proof. It is straightforward to verify using assumption (ii) in Definition 2.1 that for all $v_h \in V_h^k$,

$$\|R_h^l([\![v_h]\!])\|^2_{L^2(\Omega)^d} \leq N_\partial \sum_{F\in\mathcal{F}_h} \|r_F^l([\![v_h]\!])\|^2_{L^2(\Omega)^d}. \tag{16}$$

Furthermore, owing to the trace inequality (3) and proceeding as in [8], it is inferred that for all $F \in \mathcal{F}_h$,

$$\|r_F^l([\![v_h]\!])\|^2_{L^2(\Omega)^d} \leq c_{k,l,\mathcal{P}} \frac{1}{h_F} \int_F |[\![v_h]\!]|^2. \tag{17}$$

As a result,

$$\|R_h^l([\![v_h]\!])\|^2_{L^2(\Omega)^d} \leq c_{k,l,\mathcal{P}} |v_h|^2_{\mathrm{J},\mathcal{F}_h,-1}.$$

Using the triangle inequality yields (15). □

The main property of the discrete gradient operators defined by (14) is their weak convergence in $L^2(\Omega)^d$ when evaluated on bounded sequences in the $\|\cdot\|_{\mathrm{DG}}$-norm.

Theorem 2.2 (Compactness and weak convergence of discrete gradients)**.** *Let $k \geq 1$ and let $l \geq 0$. Let $\{v_h\}_{h\in\mathcal{H}}$ be a sequence in V_h^k. Assume that this sequence is bounded in the $\|\cdot\|_{\mathrm{DG}}$-norm. Then, there exists a function $v \in H_0^1(\Omega)$ such that as* $\mathrm{size}(\mathcal{T}_h) \to 0$*, up to a subsequence, $v_h \to v$ strongly in $L^2(\Omega)$ and for all $l \geq 0$, $G_h^l(v_h) \rightharpoonup \nabla v$ weakly in $L^2(\Omega)^d$.*

Proof. Owing to Theorem 6.2 applied with $p = 2$ and extending the functions v_h by zero outside Ω, there exists a subsequence still denoted $\{v_h\}_{h\in\mathcal{H}}$ and a function $v \in L^2(\mathbb{R}^d)$ such that as $\mathrm{size}(\mathcal{T}_h) \to 0$, $v_h \to v$ strongly in $L^2(\mathbb{R}^d)$. Moreover, since $\{G_h^l(v_h)\}_{h\in\mathcal{H}}$ is bounded in $L^2(\mathbb{R}^d)^d$ owing to Proposition 2.1, up to a new subsequence, there is $w \in L^2(\mathbb{R}^d)^d$ s.t. $G_h^l(v_h) \rightharpoonup w$ weakly in $L^2(\mathbb{R}^d)^d$. To prove that $w = \nabla v$, let $\varphi \in C_c^\infty(\mathbb{R}^d)^d$ and observe that

$$\begin{aligned}\int_{\mathbb{R}^d} G_h^l(v_h)\cdot\varphi &= -\int_{\mathbb{R}^d} v_h(\nabla\cdot\varphi) - \int_{\mathbb{R}^d} R_h^l([\![v_h]\!])\cdot(\varphi - \pi_h^0\varphi) + \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\varphi - \pi_h^0\varphi\}\!\}\cdot\nu_F[\![v_h]\!] \\ &= T_1 + T_2 + T_3.\end{aligned}$$

Letting $\mathrm{size}(\mathcal{T}_h) \to 0$, we observe that $T_1 \to -\int_{\mathbb{R}^d} v(\nabla\cdot\varphi)$ and that $T_2 \to 0$ since $\|\varphi - \pi_h^0\varphi\|_{L^\infty(\mathbb{R}^d)^d} \to 0$ and $\{R_h^l([\![v_h]\!])\}_{h\in\mathcal{H}}$ is bounded in $L^2(\mathbb{R}^d)^d$. Furthermore, the Cauchy–Schwarz inequality, together with assumption (iii) in Definition 2.1, yields

$$|T_3| \leq C\|\varphi - \pi_h^0\varphi\|_{L^\infty(\mathbb{R}^d)^d}|v_h|_{\mathrm{J},\mathcal{F}_h,-1} \leq C'\|\varphi - \pi_h^0\varphi\|_{L^\infty(\mathbb{R}^d)^d},$$

which tends to zero as $\mathrm{size}(\mathcal{T}_h) \to 0$ owing to (9). As a result,

$$\int_{\mathbb{R}^d} w\cdot\varphi = \lim_{\mathrm{size}(\mathcal{T}_h)\to 0} \int_{\mathbb{R}^d} G_h^l(v_h)\cdot\varphi = -\int_{\mathbb{R}^d} v(\nabla\cdot\varphi),$$

implying that $w = \nabla v$. Hence, $v \in H^1(\mathbb{R}^d)$ and since v is zero outside Ω, v is in $H_0^1(\Omega)$. □

It is useful to introduce for all $l \geq 0$, further discrete gradient operators $\mathcal{G}_h^l : V_h^k \to [V_h^{\max(k-1,l)}]^d$ s.t.

$$\forall v_h \in V_h^k, \qquad \mathcal{G}_h^l(v_h) \stackrel{\mathrm{def}}{=} \nabla_h v_h - \sum_{F\in\mathcal{F}_h^i} r_F^l([\![v_h]\!]). \tag{18}$$

The difference with the discrete gradient operator G_h^l defined by (14) is that boundary faces are not included in (18). Clearly, the discrete gradient operators $\mathcal{G}_h^l$ also satisfy the stability property (15). More importantly, these operators also satisfy the conclusions of Theorem 2.2. This is so because φ in the above proof is compactly supported; hence, as size$(\mathcal{T}_h) \to 0$, the mesh becomes fine enough so that all the mesh elements having a boundary face are located outside the support of φ.

3. The Poisson problem

Let $f \in L^r(\Omega)$ with $r = \frac{2d}{d+2}$ if $d \geq 3$ and $r > 1$ if $d = 2$. Set $r' \stackrel{\text{def}}{=} \frac{r}{r-1}$. Consider the following model problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{19}$$

The weak formulation of this problem consists of finding $u \in H_0^1(\Omega)$ s.t. for all $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v. \tag{20}$$

It is well-known that this problem is well–posed. In particular, owing to the Sobolev embedding $\|v\|_{L^{r'}(\Omega)} \leq \mathcal{S}_{2,r'} \|\nabla v\|_{L^2(\Omega)^d}$ valid for all $v \in H_0^1(\Omega)$, and using Hölder's inequality, it is inferred that

$$\|\nabla u\|_{L^2(\Omega)^d}^2 = \int_\Omega f u \leq \|f\|_{L^r(\Omega)} \|u\|_{L^{r'}(\Omega)} \leq \mathcal{S}_{2,r'} \|f\|_{L^r(\Omega)} \|\nabla u\|_{L^2(\Omega)^d}, \tag{21}$$

yielding the a priori bound $\|\nabla u\|_{L^2(\Omega)^d} \leq \mathcal{S}_{2,r'} \|f\|_{L^r(\Omega)}$.

3.1. **Symmetric formulation.** Let $k \geq 1$. For the sake of simplicity, discrete gradients are built using the lifting operators r_F^k (see Remark 3.2 for further discussion) and to alleviate the notation, the superscript k is omitted. This convention is kept for the rest of this work. For all $(v_h, w_h) \in V_h^k \times V_h^k$, consider the following symmetric DG bilinear form

$$a_h(v_h, w_h) \stackrel{\text{def}}{=} \int_\Omega G_h(v_h) \cdot G_h(w_h) + j_h(v_h, w_h), \tag{22}$$

with the stabilization bilinear form

$$j_h(v_h, w_h) \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h} \eta \int_\Omega r_F([\![v_h]\!]) \cdot r_F([\![w_h]\!]) - \int_\Omega R_h([\![v_h]\!]) \cdot R_h([\![w_h]\!]), \tag{23}$$

where $\eta \in \mathbb{R}_+$ is a penalty parameter. Henceforth, we assume that

$$\eta > N_\partial. \tag{24}$$

Lemma 3.1 (Continuity). *For all $(v, w) \in H_+^1(\Omega) \times H_+^1(\Omega)$,*

$$a_h(v, w) \leq c_{\eta,k,\mathcal{P}} \|v\|_{\mathrm{DG}} \|w\|_{\mathrm{DG}}. \tag{25}$$

Proof. For all $(v, w) \in H_+^1(\Omega) \times H_+^1(\Omega)$, the Cauchy–Schwarz inequality yields

$$a_h(v, w)^2 \leq (\|G_h(v)\|_{L^2(\Omega)^d}^2 + j_h(v, v))(\|G_h(w)\|_{L^2(\Omega)^d}^2 + j_h(w, w)).$$

Clearly, for $v \in H_+^1(\Omega)$ such that $v = v_1 + v_h$ with $v_1 \in H_0^1(\Omega)$ and $v_h \in V_h^k$, $\|G_h(v)\|_{L^2(\Omega)^d} \leq \|\nabla_h v\|_{L^2(\Omega)^d} + \|R_h(v_h)\|_{L^2(\Omega)^d}$ since $R_h(v_1) = 0$. Proposition 2.1

then yields $\|R_h(v_h)\|_{L^2(\Omega)^d} \le c_{k,\mathcal{P}}|v_h|_{\mathrm{J},\mathcal{F}_h,-1} = c_{k,\mathcal{P}}|v|_{\mathrm{J},\mathcal{F}_h,-1}$ since $|v_1|_{\mathrm{J},\mathcal{F}_h,-1} = 0$. As a result,

$$\|G_h(v)\|_{L^2(\Omega)^d} \le c_{k,\mathcal{P}}\|v\|_{\mathrm{DG}},$$

and similarly, $j_h(v,v) \le (N_\partial + \eta)c_{k,\mathcal{P}}|v|^2_{\mathrm{J},\mathcal{F}_h,-1}$, completing the proof. □

Lemma 3.2 (Coercivity). *For all $v_h \in V_h^k$,*

$$\|G_h(v_h)\|^2_{L^2(\Omega)^d} + (\eta - N_\partial)\sum_{F\in\mathcal{F}_h} \|r_F([\![v_h]\!])\|^2_{L^2(\Omega)^d} \le a_h(v_h, v_h). \tag{26}$$

Furthermore, there is $\alpha > 0$, depending on η, k and $\mathcal{P}$ such that for all $v_h \in V_h^k$,

$$\alpha\|v_h\|^2_{\mathrm{DG}} \le a_h(v_h, v_h). \tag{27}$$

Proof. Estimate (26) directly results from (16). To verify (27), observe first that proceeding as in [8] using assumptions (iv) and (vi) in Definition 2.1 yields for all $F \in \mathcal{F}_h$,

$$\frac{1}{h_F}\int_F |[\![v_h]\!]|^2 \le c'_{k,\mathcal{P}}\|r_F([\![v_h]\!])\|^2_{L^2(\Omega)^d}. \tag{28}$$

Using the triangle inequality, it is then inferred that

$$\begin{aligned}\|v_h\|^2_{\mathrm{DG}} &\le 2\|G_h(v_h)\|^2_{L^2(\Omega)^d} + 2\|R_h([\![v_h]\!])\|^2_{L^2(\Omega)^d} + |v_h|^2_{\mathrm{J},\mathcal{F}_h,-1}\\ &\le 2\|G_h(v_h)\|^2_{L^2(\Omega)^d} + (2N_\partial + (c'_{k,\mathcal{P}})^{-1})\textstyle\sum_{F\in\mathcal{F}_h}\|r_F([\![v_h]\!])\|^2_{L^2(\Omega)^d}\\ &\le \max(2, (2N_\partial + (c'_{k,\mathcal{P}})^{-1})(\eta - N_\partial)^{-1})a_h(v_h, v_h),\end{aligned}$$

the last inequality resulting from (26). □

Remark 3.1. A straightforward calculation shows that

$$\begin{aligned}a_h(v_h, w_h) = \int_\Omega \nabla_h v_h\cdot\nabla_h w_h - \sum_{F\in\mathcal{F}_h}\int_F \left(\nu_F\cdot\{\!\{\nabla_h v_h\}\!\}[\![w_h]\!] + \nu_F\cdot\{\!\{\nabla_h w_h\}\!\}[\![v_h]\!]\right)\\ + \sum_{F\in\mathcal{F}_h}\eta\int_\Omega r_F([\![v_h]\!])\cdot r_F([\![w_h]\!]),\end{aligned}$$

yielding the IP-type method introduced in [4]. It is also possible to consider the stabilization bilinear form

$$j_h^{\mathrm{SIPG}}(v_h, w_h) \stackrel{\mathrm{def}}{=} \sum_{F\in\mathcal{F}_h}\eta\frac{1}{h_F}\int_F [\![v_h]\!][\![w_h]\!] - \int_\Omega R_h([\![v_h]\!])\cdot R_h([\![w_h]\!]),$$

yielding the usual Symmetric Interior Penalty method (SIPG) [1]. In this case, the minimal threshold for the penalty parameter η depends on the constant in the trace inequality (3). It is also possible to consider the stabilization bilinear form

$$j_h^{\mathrm{LDG}}(v_h, w_h) \stackrel{\mathrm{def}}{=} \sum_{F\in\mathcal{F}_h}\eta\frac{1}{h_F}\int_F [\![v_h]\!][\![w_h]\!],$$

yielding the usual Local Discontinuous Galerkin method (LDG) [11]. The advantage is that the parameter η needs only be positive, the disadvantage is however that the stencil is enlarged to neighbors of neighbors. Moreover, working with any of the two above stabilization bilinear forms allows to lift assumption (vi) in Definition 2.1.

For all $h \in \mathcal{H}$, Lemma 3.2 implies that there is a unique $u_h \in V_h^k$ s.t.

$$a_h(u_h, v_h) = \int_\Omega f v_h, \qquad \forall v_h \in V_h^k. \tag{29}$$

Theorem 3.1 (Convergence for Poisson problem). *Let $\{u_h\}_{h\in\mathcal{H}}$ be the sequence of approximate solutions generated by solving the discrete problems* (29) *on the admissible meshes $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Then, as* $\mathrm{size}(\mathcal{T}_h) \to 0$,

$$u_h \to u, \qquad \text{in } L^2(\Omega), \tag{30}$$

$$\nabla_h u_h \to \nabla u, \quad \text{in } L^2(\Omega)^d, \tag{31}$$

$$|u_h|_{\mathrm{J},\mathcal{F}_h,-1} \to 0, \tag{32}$$

where $u \in H_0^1(\Omega)$ is the unique solution to (19).

Proof. (i) A priori estimate. Using Lemma 3.2 and Hölder's inequality, it is inferred that

$$\alpha \|u_h\|_{\mathrm{DG}}^2 \leq a(u_h, u_h) = \int_\Omega f u_h \leq \|f\|_{L^r(\Omega)} \|u_h\|_{L^{r'}(\Omega)}.$$

Hence, owing to Theorem 2.1, the sequence $\{u_h\}_{h\in\mathcal{H}}$ is bounded in the $\|\cdot\|_{\mathrm{DG}}$-norm.
(ii) L^2-convergence of a subsequence, regularity of the limit and weak convergence of discrete gradient. Owing to Theorem 2.2, there exists $u \in H_0^1(\Omega)$ such that, as $\mathrm{size}(\mathcal{T}_h) \to 0$, up to a subsequence, $u_h \to u$ strongly in $L^2(\Omega)$ and $G_h(u_h) \rightharpoonup \nabla u$ weakly in $L^2(\Omega)^d$.
(iii) Identification of u and convergence of the whole sequence. Let us first prove that for all $\varphi \in C_c^\infty(\Omega)$,

$$a_h(u_h, \pi_h\varphi) \to \int_\Omega \nabla u \cdot \nabla \varphi. \tag{33}$$

Indeed, observe that

$$a_h(u_h, \pi_h\varphi) = a_h(u_h, \pi_h\varphi - \varphi) + \int_\Omega G_h(u_h) \cdot \nabla\varphi = T_1 + T_2.$$

Clearly, $T_1 \to 0$ owing to Lemma 3.1 since $\|u_h\|_{\mathrm{DG}}$ is bounded and $\|\varphi - \pi_h\varphi\|_{\mathrm{DG}}$ converges to zero. Furthermore, $T_2 \to \int_\Omega \nabla u \cdot \nabla\varphi$ owing to the weak convergence of the discrete gradient. A direct consequence of (33) is that for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega f\varphi \leftarrow \int_\Omega f\pi_h\varphi = a_h(u_h, \pi_h\varphi) \to \int_\Omega \nabla u \cdot \nabla\varphi.$$

Thus, u solves the Poisson problem by density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$. Since the solution to this problem is unique, the whole sequence $\{u_h\}_{h\in\mathcal{H}}$ strongly converges to u in $L^2(\Omega)$ and $\{G_h(u_h)\}_{h\in\mathcal{H}}$ weakly converges to ∇u in $L^2(\Omega)^d$.
(iv) Strong convergence of the discrete gradient and of the jumps. Owing to (26) and to weak convergence,

$$\liminf a_h(u_h, u_h) \geq \liminf \|G_h(u_h)\|_{L^2(\Omega)^d}^2 \geq \|\nabla u\|_{L^2(\Omega)^d}^2.$$

Furthermore, still owing to (26),

$$\|G_h(u_h)\|_{L^2(\Omega)^d}^2 \leq a_h(u_h, u_h) = \int_\Omega f u_h,$$

yielding

$$\begin{aligned}\limsup \|G_h(u_h)\|^2_{L^2(\Omega)^d} &\leq \limsup a_h(u_h,u_h)\\ &= \limsup \int_\Omega f u_h = \int_\Omega f u = \|\nabla u\|^2_{L^2(\Omega)^d}.\end{aligned}$$

Thus, $\|G_h(u_h)\|_{L^2(\Omega)^d} \to \|\nabla u\|_{L^2(\Omega)^d}$, classically yielding the strong convergence of the discrete gradient in $L^2(\Omega)^d$. Note that $a_h(u_h,u_h) \to \|\nabla u\|^2_{L^2(\Omega)^d}$ also. Finally, owing to (26),

$$(\eta - N_\partial) \sum_{F\in\mathcal{F}_h} \|r_F(\llbracket u_h \rrbracket)\|^2_{L^2(\Omega)^d} \leq a_h(u_h,u_h) - \|G_h(u_h)\|^2_{L^2(\Omega)^d},$$

and since $\eta > N_\partial$ and the right-hand side tends to zero, it is inferred using (28) that $|u_h|_{\mathrm{J},\mathcal{F}_h,-1} \to 0$. Moreover, using (16) to estimate the second term yields

$$\|\nabla_h u_h - \nabla u\|_{L^2(\Omega)^d} \leq \|G_h(u_h) - \nabla u\|_{L^2(\Omega)^d} + \|R_h(u_h)\|_{L^2(\Omega)^d} \to 0,$$

as $\mathrm{size}(\mathcal{T}_h) \to 0$, concluding the proof. □

Remark 3.2. To prove the convergence of the method, it is sufficient to work with the lifting operators r_F^0. However, if the exact solution u turns out to be more regular, optimal-order convergence rates can be established in the $\|\cdot\|_{\mathrm{DG}}$-norm when working with the lifting operators r_F^{k-1} or r_F^k. (The latter choice may be preferable for implementation purposes, especially if non-hierarchical, e.g. nodal-based, basis functions are used.) For instance, if u belongs to the broken Sobolev space $H^{k+1}(\mathcal{T}_h)$, the usual a priori error analysis techniques can be used to infer a bound of the form $\|u - u_h\|_{\mathrm{DG}} \leq c_u \mathrm{size}(\mathcal{T}_h)^k$. A minor difference with the somewhat more usual formulation which does not employ explicitly the discrete gradient operators, is that (29) is only weakly consistent, but not strongly consistent. Indeed, it is easily seen that for all $v_h \in V_h^k$,

$$\begin{aligned}a_h(u_h - u, v_h) &= \sum_{F\in\mathcal{F}_h} \int_F \nu_F \cdot \{\!\{\pi_h^k(\nabla u) - \nabla u\}\!\} \llbracket v_h \rrbracket\\ &\leq c_u \mathrm{size}(\mathcal{T}_h)^k \|v_h\|_{\mathrm{DG}}.\end{aligned}$$

3.2. **Nonsymmetric formulations.** Nonsymmetric DG approximations to the Poisson problem (and other selfadjoint PDE's) have received some interest in the literature. Such formulations use a nonsymmetric bilinear form that can be cast into the generic form

$$a_h(v_h, w_h) = \int_\Omega \widehat{G}_h(v_h)\cdot G_h(w_h) + j_h'(v_h,w_h), \tag{34}$$

where $\widehat{G}_h$ and G_h are discrete gradient operators and where the stabilization bilinear form j_h' can differ from that given by (23). The following design conditions must be satisfied.

(i) Control on discrete gradients: there is c s.t. for all $v \in H^1_+(\Omega)$,

$$\|\widehat{G}_h(v)\|_{L^2(\Omega)^d} + \|G_h(v)\|_{L^2(\Omega)^d} \leq c\|v\|_{\mathrm{DG}}; \tag{35}$$

(ii) Strong consistency of the discrete gradient $\widehat{G}_h$ for smooth functions: for all $\varphi \in C_c^\infty(\Omega)$, $\widehat{G}_h(\varphi) = \nabla\varphi$;

(iii) Weak consistency of the discrete gradient G_h: for any sequence $\{v_h\}_{h\in\mathcal{H}}$ converging in $L^2(\Omega)$ to a function $v \in H^1_0(\Omega)$ and such that $\|G_h(v_h)\|_{L^2(\Omega)^d}$ is bounded, up to a subsequence, $G_h(v_h) \rightharpoonup \nabla v$ weakly in $L^2(\Omega)^d$;

(iv) Stabilization: the bilinear form j'_h is symmetric and positive, there is c s.t. for all $v \in H^1_+(\Omega)$, $j_h(v,v) \le c|v|^2_{\mathrm{J},\mathcal{F}_h,-1}$ and there is $\eta_* > 0$ such that for all $v_h \in V^k_h$,

$$a_h(v_h, v_h) \ge \eta_* \|v_h\|^2_{\mathrm{DG}}. \tag{36}$$

Observe that the continuity property of j_h implies that for all $\varphi \in C^\infty_c(\Omega)$, $j_h(\varphi,\cdot) = 0$ and that (36) implies that the discrete problem (34) is well–posed.

Under the above assumptions, the convergence of the sequence of discrete DG approximations can be proven. The proof, however, proceeds along a slightly different path with respect to the symmetric formulation.

Theorem 3.2. *Let $\{u_h\}_{h\in\mathcal{H}}$ be the sequence of approximate solutions generated by solving the discrete problems* (29) *with the bilinear form a_h given by* (34) *on the admissible meshes $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Assume the above design conditions* (i)–(iv)*. Then, as* $\mathrm{size}(\mathcal{T}_h) \to 0$*, $u_h \to u$ in $L^2(\Omega)$ and $\widehat{G}_h(u_h) \to \nabla u$ in $L^2(\Omega)^d$.*

Proof. (i) Proceeding as before, it is inferred from (iv) that the sequence $\{u_h\}_{h\in\mathcal{H}}$ is bounded in the $\|\cdot\|_{\mathrm{DG}}$-norm so that there exists $u \in H^1_0(\Omega)$ such that, up to a subsequence, $u_h \to u$ in $L^2(\Omega)$ as $\mathrm{size}(\mathcal{T}_h) \to 0$. Furthermore, since the sequence $\{G_h(u_h)\}_{h\in\mathcal{H}}$ is bounded in $L^2(\Omega)^d$ owing to (i), the weak consistency property (iii) yields that (up to a new subsequence) $G_h(u_h)$ weakly converges to ∇u in $L^2(\Omega)^d$.
(ii) Strong convergence of $\widehat{G}_h(u_h) \in L^2(\Omega)^d$. Let $\varphi \in C^\infty_c(\Omega)$. Observe that

$$\begin{aligned}\tfrac{1}{2}\|\widehat{G}_h(u_h) - \nabla u\|^2_{L^2(\Omega)^d} &\le \|\widehat{G}_h(u_h) - \widehat{G}_h(\pi_h\varphi)\|^2_{L^2(\Omega)^d} + \|\widehat{G}_h(\pi_h\varphi) - \nabla u\|^2_{L^2(\Omega)^d}\\ &= T_1 + T_2.\end{aligned}$$

Clearly, as $\mathrm{size}(\mathcal{T}_h) \to 0$, $T_2 \to \|\nabla(\varphi-u)\|^2_{L^2(\Omega)^d}$ since, using (i) and (ii), $\widehat{G}_h(\pi_h\varphi) - \nabla\varphi = \widehat{G}_h(\pi_h\varphi - \varphi)$ is bounded in $L^2(\Omega)^d$ by $\|\varphi - \pi_h\varphi\|_{\mathrm{DG}}$ which converges to zero. To bound T_1, use (i) and (iv) to infer

$$\begin{aligned}T_1 = \|\widehat{G}_h(u_h - \pi_h\varphi)\|^2_{L^2(\Omega)^d} &\le \tfrac{c}{\eta_*} a_h(u_h - \pi_h\varphi, u_h - \pi_h\varphi)\\ &= \tfrac{c}{\eta_*}\left(\int_\Omega f(u_h - \pi_h\varphi) - a_h(\pi_h\varphi, u_h - \pi_h\varphi)\right)\\ &= \tfrac{c}{\eta_*}(T_{1,1} - T_{1,2}).\end{aligned}$$

Clearly, as $\mathrm{size}(\mathcal{T}_h) \to 0$, $T_{1,1} \to \int_\Omega f(u-\varphi)$. Moreover, by definition,

$$T_{1,2} = \int_\Omega \widehat{G}_h(\pi_h\varphi)\cdot G_h(u_h - \pi_h\varphi) + j'_h(\pi_h\varphi, u_h - \pi_h\varphi).$$

Since $\widehat{G}_h(\pi_h\varphi)$ strongly converges to $\nabla\varphi$ in $L^2(\Omega)^d$ and $G_h(u_h - \pi_h\varphi)$ weakly converges to $\nabla(u-\varphi)$ in $L^2(\Omega)^d$, the first term in the right-hand side converges to $\int_\Omega \nabla\varphi\cdot\nabla(u-\varphi)$. In addition, the second term is equal to $j'_h(\pi_h\varphi - \varphi, u_h - \pi_h\varphi)$ which converges to zero. Collecting the above bounds, it is inferred that

$$\limsup \|\widehat{G}_h(u_h) - \nabla u\|^2_{L^2(\Omega)^d} \le C\|u - \varphi\|^2_{H^1(\Omega)}.$$

Letting $\varphi \in C^\infty_c(\Omega)$ tend to u in $H^1_0(\Omega)$, the upper bound can be made as small as desired. This implies the strong convergence of $\widehat{G}_h(u_h)$ to ∇u in $L^2(\Omega)^d$.

(iv) Identification of the limit and convergence of the whole sequence. Let $\varphi \in C_c^\infty(\Omega)$. It is clear that as size$(\mathcal{T}_h) \to 0$, $\int_\Omega f\pi_h\varphi \to \int_\Omega f\varphi$. Furthermore,

$$a_h(u_h, \pi_h\varphi) = \int_\Omega \widehat{G}(u_h)\cdot G(\pi_h\varphi) + j_h'(u_h, \pi_h\varphi) = T_3 + T_4.$$

Clearly, $T_3 \to \int_\Omega \nabla u\cdot\nabla\varphi$ because of the strong convergence of $\widehat{G}_h(u_h)$ to ∇u in $L^2(\Omega)^d$ and the weak convergence of $G_h(\pi_h\varphi)$ to $\nabla\varphi$ in $L^2(\Omega)^d$. In addition, $|T_4| \leq c|u_h|_{\mathrm{J},\mathcal{F}_h,-1}|\varphi - \pi_h\varphi|_{\mathrm{J},\mathcal{F}_h,-1} \leq c'|\varphi - \pi_h\varphi|_{\mathrm{J},\mathcal{F}_h,-1}$ which converges to zero. As a result,

$$a_h(u_h, \pi_h\varphi) \to \int_\Omega \nabla u\cdot\nabla\varphi.$$

The proof can now be completed as in the symmetric case. □

Classical examples of the situation analyzed by Theorem 3.2 are the so-called Incomplete Interior Penalty method (IIPG) for which

$$\widehat{G}_h(v_h) = \nabla_h v_h, \tag{37}$$

and the so-called Nonsymmetric Interior Penalty method (NIPG) for which

$$\widehat{G}_h(v_h) = \nabla_h v_h + R_h([\![v_h]\!]), \tag{38}$$

together with $G_h(v_h) = \nabla_h v_h - R_h([\![v_h]\!])$ in both cases.

4. The Stokes equations

Let $f \in L^r(\Omega)^d$ with $r = \frac{2d}{d+2}$ if $d \geq 3$ and $r > 1$ if $d = 2$. Let $\nu > 0$. The components in the Cartesian basis $(e_1, \dots, e_d)$ of $\mathbb{R}^d$ of a function, say v, with values in $\mathbb{R}^d$ will be denoted by $(v_i)_{1\leq i\leq d}$. Implicit summation convention of repeated indices is adopted henceforth. Consider the Stokes equations

$$\begin{cases} -\nu\Delta u_i + \partial_i p = f_i, & \text{in } \Omega, \quad i \in \{1, \dots, d\}, \\ \partial_i u_i = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \int_\Omega p = 0. \end{cases} \tag{39}$$

The weak formulation of this system consists of finding $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ s.t. for all $(v, q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$,

$$\nu\int_\Omega \partial_j u_i \partial_j v_i - \int_\Omega p\partial_i v_i + \int_\Omega q\partial_i u_i = \int_\Omega f_i v_i. \tag{40}$$

The well–posedness of the above problem is a classical result (see e.g. [15] and references therein).

To formulate a DG approximation, we consider for each component of the velocity the symmetric DG bilinear form a_h defined by (22) and the stabilization bilinear form j_h defined by (23). For the sake of simplicity, in particular with an eye towards ease of implementation, we will consider the case of equal-order polynomial interpolation for the velocity and for the pressure. Letting $k \geq 1$, we thus set

$$U_h \stackrel{\text{def}}{=} [V_h^k]^d, \qquad P_h \stackrel{\text{def}}{=} V_h^k/\mathbb{R}, \qquad X_h \stackrel{\text{def}}{=} U_h \times P_h. \tag{41}$$

For $\mathbb{R}^d$-valued functions such as velocities, the seminorm $|\cdot|_{\mathrm{J},\mathcal{F}_h,-1}$ and the norm $\|\cdot\|_{\mathrm{DG}}$ are defined as the square root of the sum of the squares of the corresponding seminorm or norm for all the components.

4.1. **Discrete divergence operators.** Define on $U_h \times P_h$ the bilinear form

$$b_h(v_h, q_h) \stackrel{\text{def}}{=} \int_\Omega v_h\cdot\nabla_h q_h - \sum_{F\in\mathcal{F}_h^i} \int_F \nu_F\cdot\{\!\{v_h\}\!\}[\![q_h]\!]. \tag{42}$$

Integration by parts readily yields the following equivalent expression

$$b_h(v_h, q_h) = -\int_\Omega q_h \nabla_h\cdot v_h + \sum_{F\in\mathcal{F}_h} \int_F \nu_F\cdot[\![v_h]\!]\{\!\{q_h\}\!\}. \tag{43}$$

Here, $\nabla_h\cdot$ denotes the broken divergence operator acting elementwise. Furthermore, define on $P_h \times P_h$ the pressure stabilization bilinear form

$$s_h(q_h, r_h) \stackrel{\text{def}}{=} \sum_{F\in\mathcal{F}_h^i} \gamma h_F \int_F [\![q_h]\!][\![r_h]\!]. \tag{44}$$

Here, $\gamma \in \mathbb{R}_+$ is a penalty parameter. For simplicity, it will be taken equal to 1 in what follows. The basic stability result for the bilinear form b_h is the following.

Lemma 4.1. *There is $\beta > 0$, depending on Ω, k and $\mathcal{P}$, such that*

$$\forall q_h \in P_h, \qquad \beta\|q_h\|_{L^2(\Omega)} \le \sup_{0\ne v_h\in U_h} \frac{b_h(v_h, q_h)}{\|v_h\|_{\mathrm{DG}}} + |q_h|_{\mathrm{J},\mathcal{F}_h^i,1}. \tag{45}$$

Proof. Let $q_h \in P_h$. Owing to a result by Nečas [31], there is $v \in H_0^1(\Omega)^d$ s.t. $\nabla\cdot v = q_h$ and $\|v\|_{H^1(\Omega)^d} \le c_\Omega\|q_h\|_{L^2(\Omega)}$. Then,

$$\begin{aligned}
\|q_h\|_{L^2(\Omega)}^2 = \int_\Omega q_h(\nabla\cdot v) &= -\int_\Omega \nabla_h q_h\cdot v + \sum_{F\in\mathcal{F}_h^i}\int_F [\![q_h]\!]\{\!\{v\}\!\}\cdot\nu_F \\
&= -\int_\Omega \nabla_h q_h\cdot \pi_h^k v + \sum_{F\in\mathcal{F}_h^i}\int_F [\![q_h]\!]\{\!\{v\}\!\}\cdot\nu_F \\
&= -b_h(\pi_h^k v, q_h) + \sum_{F\in\mathcal{F}_h^i}\int_F [\![q_h]\!]\{\!\{v - \pi_h^k v\}\!\}\cdot\nu_F \\
&= T_1 + T_2.
\end{aligned}$$

Since $\|\pi_h^k v\|_{\mathrm{DG}} \le c_{k,\mathcal{P}}\|v\|_{H^1(\Omega)^d} \le c_{\Omega,k,\mathcal{P}}\|q_h\|_{L^2(\Omega)}$ because of (10), it is inferred that

$$|T_1| \le \frac{|b_h(\pi_h^k v, q_h)|}{\|\pi_h^k v\|_{\mathrm{DG}}}\|\pi_h^k v\|_{\mathrm{DG}} \le c_{\Omega,k,\mathcal{P}} \left(\sup_{0\ne v_h\in U_h} \frac{b_h(v_h, q_h)}{\|v_h\|_{\mathrm{DG}}}\right)\|q_h\|_{L^2(\Omega)}.$$

Similarly, $|T_2| \le c_{\Omega,k,\mathcal{P}}|q_h|_{\mathrm{J},\mathcal{F}_h^i,1}\|q_h\|_{L^2(\Omega)}$, whence the conclusion follows. □

Recall the discrete gradient operators G_h^l and $\mathcal{G}_h^l$ defined in §2.3. For all $l \ge 0$, introduce now the discrete divergence operators $D_h^l : U_h \to V_h^{\max(k-1,l)}$ defined s.t.

$$\forall v_h \in U_h, \qquad D_h^l(v_h) \stackrel{\text{def}}{=} G_h^l(v_{h,j})\cdot e_j. \tag{46}$$

For $l \ge k$, the following integration by parts formula holds for all $(v_h, q_h) \in X_h$:

$$\int_\Omega q_h D_h^l(v_h) + \int_\Omega \mathcal{G}_h^l(q_h)\cdot v_h = 0. \tag{47}$$

Moreover, it is easily seen that for $l \geq k$ and for all $(v_h, q_h) \in X_h$,

$$b_h(v_h, q_h) = \int_\Omega v_h \cdot \mathcal{G}_h^l(q_h) = -\int_\Omega q_h D_h^l(v_h). \tag{48}$$

As before, superscripts will be dropped if $l = k$.

4.2. **Stability estimates and discrete well–posedness.** For all $((u_h, p_h), (v_h, q_h)) \in X_h \times X_h$, define the bilinear form

$$l_h((u_h, p_h), (v_h, q_h)) \stackrel{\text{def}}{=} \nu a_h(u_{h,i}, u_{h,i}) + b_h(v_h, p_h) - b_h(u_h, q_h) + s_h(p_h, q_h). \tag{49}$$

The discrete Stokes equations consists of seeking $(u_h, p_h) \in X_h$ s.t.

$$l_h((u_h, p_h), (v_h, q_h)) = \int_\Omega f_i v_{h,i}, \qquad \forall (v_h, q_h) \in X_h. \tag{50}$$

Define the following norm

$$\|(v_h, q_h)\|_{\mathrm{S}}^2 \stackrel{\text{def}}{=} \|v_h\|_{\mathrm{DG}}^2 + |q_h|_{\mathrm{J},\mathcal{F}_h^i,1}^2 + \|q_h\|_{L^2(\Omega)}^2. \tag{51}$$

A direct consequence of (27) applied componentwise is the following result:

Lemma 4.2. *Let $\alpha > 0$ as in Lemma 3.2. Then, the following holds:*

$$\forall (v_h, q_h) \in X_h, \qquad \nu\alpha \|v_h\|_{\mathrm{DG}}^2 + |q_h|_{\mathrm{J},\mathcal{F}_h^i,1}^2 \leq l_h((v_h, q_h), (v_h, q_h)). \tag{52}$$

Combining Lemmata 4.1 and 4.2 yields the following stability result.

Lemma 4.3. *There is $c_l > 0$ depending on ν, k, $\mathcal{P}$, Ω and η s.t.*

$$\forall (v_h, q_h) \in X_h, \qquad c_l \|v_h, q_h)\|_{\mathrm{S}} \leq \sup_{0 \neq (w_h, r_h) \in X_h} \frac{l_h((v_h, q_h), (w_h, r_h))}{\|(w_h, r_h)\|_{\mathrm{S}}}. \tag{53}$$

Proof. Let $(v_h, q_h) \in X_h$ and set $\mathbb{S} \stackrel{\text{def}}{=} \sup_{0 \neq (w_h, r_h) \in X_h} \frac{l_h((v_h, q_h), (w_h, r_h))}{\|(w_h, r_h)\|_{\mathrm{S}}}$. Owing to Lemma 4.2,

$$\nu\alpha \|v_h\|_{\mathrm{DG}}^2 + |q_h|_{\mathrm{J},\mathcal{F}_h^i,1}^2 \leq \mathbb{S} \|(v_h, q_h)\|_{\mathrm{S}},$$

and it only remains to control $\|q_h\|_{L^2(\Omega)}^2$. Using Lemmata 4.1 and 3.1 yields

$$\begin{aligned}
\beta \|q_h\|_{L^2(\Omega)} &\leq \sup_{0 \neq w_h \in U_h} \frac{b_h(w_h, q_h)}{\|w_h\|_{\mathrm{DG}}} + |q_h|_{\mathrm{J},\mathcal{F}_h^i,1} \\
&\leq \sup_{0 \neq w_h \in U_h} \frac{\nu a_h(v_{h,i}, w_{h,i})}{\|w_h\|_{\mathrm{DG}}} + \sup_{0 \neq w_h \in U_h} \frac{l_h((v_h, q_h), (w_h, 0))}{\|(w_h, 0)\|_{\mathrm{S}}} + |q_h|_{\mathrm{J},\mathcal{F}_h^i,1} \\
&\leq \nu c_{\eta,k,\mathcal{P}} \|v_h\|_{\mathrm{DG}} + \mathbb{S} + |q_h|_{\mathrm{J},\mathcal{F}_h^i,1}.
\end{aligned}$$

The conclusion is straightforward. □

A direct consequence of Lemma 4.3 is that for all $h \in \mathcal{H}$, the discrete problem (50) admits a unique solution $(u_h, p_h) \in X_h$.

4.3. **Convergence analysis.** In this section, we are now interested in the convergence of the sequence $\{(u_h, p_h)\}_{h\in\mathcal{H}}$ of solutions to the discrete Stokes equations (50) towards the unique solution (u, p) of the continuous Stokes equations (40).

Theorem 4.1 (Convergence for Stokes equations)**.** *Let $\{(u_h, p_h)\}_{h\in\mathcal{H}}$ be the sequence of approximate solutions generated by solving the discrete problems* (50) *on the admissible meshes $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Then, as* $\operatorname{size}(\mathcal{T}_h) \to 0$,

$$u_h \to u, \quad \text{in } L^2(\Omega)^d, \tag{54}$$

$$\nabla_h u_h \to \nabla u, \quad \text{in } L^2(\Omega)^{d,d}, \tag{55}$$

$$|u_h|_{\mathrm{J},\mathcal{F}_h,-1} \to 0, \tag{56}$$

$$p_h \to p, \quad \text{in } L^2(\Omega), \tag{57}$$

$$|p_h|_{\mathrm{J},\mathcal{F}_h^i,1} \to 0, \tag{58}$$

where $(u, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ is the unique solution to (40).

Proof. (i) A priori estimates. Owing to the inf-sup condition (53), the assumption on f and the discrete Sobolev embedding, the sequence $\{(u_h, p_h)\}_{h\in\mathcal{H}}$ is bounded in the $\|\cdot\|_{\mathrm{S}}$-norm. Hence, up to a subsequence, there is $(u, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ s.t. $u_h \to u$ strongly in $L^2(\Omega)^d$, $G_h(u_{h,i}) \rightharpoonup \nabla u_i$ weakly in $L^2(\Omega)^d$ for all $i \in \{1, \dots, d\}$, and $p_h \rightharpoonup p$ weakly in $L^2(\Omega)$.
(ii) Identification of the limit and convergence of the whole sequence. Let $\varphi \in C_c^\infty(\Omega)^d$. Testing with $(\pi_h\varphi, 0)$ yields

$$\nu a_h(u_{h,i}, \pi_h\varphi_i) + b_h(\pi_h\varphi, p_h) = \int_\Omega f_i \pi_h \varphi_i.$$

Clearly, as $\operatorname{size}(\mathcal{T}_h) \to 0$, the right-hand side tends to $\int_\Omega f_i\varphi_i$. Furthermore, proceeding as for the Poisson problem yields that the first term in the left-hand side converges to $\nu \int_\Omega \partial_j u_i \partial_j \varphi_i$. Consider now the second term and observe that

$$b_h(\pi_h\varphi, p_h) = -\int_\Omega p_h \nabla_h\cdot\pi_h\varphi + \sum_{F\in\mathcal{F}_h} \int_F \nu_F\cdot[\![\pi_h\varphi]\!]\{\!\{p_h\}\!\} = T_1 + T_2.$$

Owing to the weak convergence of $\{p_h\}_{h\in\mathcal{H}}$ to p in $L^2(\Omega)$ and the strong convergence of $\{\nabla_h\cdot\pi_h\varphi\}_{h\in\mathcal{H}}$ to $\nabla\cdot\varphi$ in $L^2(\Omega)$, T_1 tends to $-\int_\Omega p(\nabla\cdot\varphi)$. Moreover, using the trace inequality (3) to estimate $\|\{\!\{p_h\}\!\}\|_{L^2(F)}$ yields

$$|T_2| \le c_{k,\mathcal{P}} \|\varphi - \pi_h\varphi\|_{\mathrm{DG}} \|p_h\|_{L^2(\Omega)} \le C\|\varphi - \pi_h\varphi\|_{\mathrm{DG}}.$$

Hence, T_2 tends to zero. As a result,

$$\nu \int_\Omega \partial_j u_i \partial_j \varphi_i - \int_\Omega p\partial_j\varphi_j = \int_\Omega f_i\varphi_i.$$

Let now $\psi \in C_c^\infty(\Omega)/\mathbb{R}$. Testing with $(0, \pi_h\psi)$ yields

$$-b_h(u_h, \pi_h\psi) + s_h(p_h, \pi_h\psi) = 0.$$

Using (48) yields $-b_h(u_h, \pi_h\psi) = \int_\Omega \pi_h\psi D_h(u_h)$. Since $\{D_h(u_h)\}_{h\in\mathcal{H}}$ weakly converges to $\nabla\cdot u$ in $L^2(\Omega)$ and $\{\pi_h\psi\}_{h\in\mathcal{H}}$ strongly converges to ψ in $L^2(\Omega)$, the first term in the left-hand side tends to $\int_\Omega \psi(\nabla\cdot u)$. Furthermore, the second term tends to zero since

$$|s_h(p_h, \pi_h\psi)| \le c_{k,\mathcal{P}} |p_h|_{\mathrm{J},\mathcal{F}_h^i,1} |\pi_h\psi|_{\mathrm{J},\mathcal{F}_h^i,1} \le C|\pi_h\psi|_{\mathrm{J},\mathcal{F}_h^i,1},$$

and this upper bound tends to zero. Hence,

$$\int_\Omega \psi \partial_j u_j = 0.$$

By density of $C_c^\infty(\Omega)^d \times C_c^\infty(\Omega)/\mathbb{R}$ in $H_0^1(\Omega)^d \times L_0^2(\Omega)$, this shows that (u,p) solves the Stokes equations (40). Since the solution to this problem is unique, the whole sequence $\{(u_h,p_h)\}_{h\in\mathcal{H}}$ converges.

(iii) Strong convergence of the velocity gradient and convergence of velocity and pressure jumps. Observe that

$$\begin{aligned}\int_\Omega f_i u_{h,i} = l_h((u_h,p_h),(u_h,p_h)) &\geq \nu a_h(u_{h,i},u_{h,i}) + s_h(p_h,p_h) \\ &\geq \nu a_h(u_{h,i},u_{h,i}) \geq \sum_{i=1}^d \nu \|G_h(u_{h,i})\|^2_{L^2(\Omega)^d}.\end{aligned}$$

Thus,

$$\limsup \sum_{i=1}^d \nu \|G_h(u_{h,i})\|^2_{L^2(\Omega)^d} \leq \limsup \int_\Omega f_i u_{h,i} = \int_\Omega f_i u_i = \nu \|\nabla u\|^2_{L^2(\Omega)^{d,d}}.$$

Since $\liminf \sum_{i=1}^d \|G_h(u_{h,i})\|^2_{L^2(\Omega)^d} \geq \|\nabla u\|^2_{L^2(\Omega)^{d,d}}$ owing to weak convergence, this classically implies the strong convergence in $L^2(\Omega)^d$ of $G_h(u_{h,i})$ to ∇u_i for all $i \in \{1,\dots,d\}$. The above inequalities also imply that $a_h(u_{h,i},u_{h,i}) \to \|\nabla u\|^2_{L^2(\Omega)^{d,d}}$, and proceeding as for the Poisson problem, it is deduced that $|u_h|_{\mathrm{J},\mathcal{F}_h,-1} \to 0$. Finally, since

$$|p_h|^2_{\mathrm{J},\mathcal{F}_h^i,1} = b_h(u_h,p_h) = \int_\Omega f_i u_{h,i} - \nu a_h(u_{h,i},u_{h,i}),$$

it is inferred that $|p_h|_{\mathrm{J},\mathcal{F}_h^i,1} \to 0$.

(iv) Strong convergence of the pressure. Using again the result by Nečas [31], let $v(p_h) \in H_0^1(\Omega)^d$ be s.t. $\nabla\cdot v(p_h) = p_h$ with $\|v(p_h)\|_{H^1(\Omega)^d} \leq c_\Omega \|p_h\|_{L^2(\Omega)}$ and set $v_h = \pi_h^k v(p_h)$. Then, proceeding as in the proof of Lemma 4.1 yields

$$\begin{aligned}\|p_h\|^2_{L^2(\Omega)} &\leq c_{\Omega,k,\mathcal{P}} |p_h|_{\mathrm{J},\mathcal{F}_h^i,1} \|p_h\|_{L^2(\Omega)} - b_h(v_h,p_h) \\ &\leq c_{\Omega,k,\mathcal{P}} |p_h|_{\mathrm{J},\mathcal{F}_h^i,1} \|p_h\|_{L^2(\Omega)} + \nu a_h(u_{h,i},v_{h,i}) - \int_\Omega f_i v_{h,i} = T_1 + T_2 - T_3.\end{aligned}$$

Since $|p_h|_{\mathrm{J},\mathcal{F}_h^i,1}$ tends to zero and $\|p_h\|_{L^2(\Omega)}$ is bounded, T_1 converges to zero. Furthermore, since the sequence $\{v_h\}_{h\in\mathcal{H}}$ is bounded in the $\|\cdot\|_{\mathrm{DG}}$-norm because $\|v_h\|_{\mathrm{DG}} \leq c_{k,\mathcal{P}} \|v(p_h)\|_{H^1(\Omega)^d} \leq c_{\Omega,k,\mathcal{P}} \|p_h\|_{L^2(\Omega)}$, there is $v \in H_0^1(\Omega)^d$ such that, up to a subsequence, $v_h \to v$ strongly in $L^2(\Omega)^d$ and $G_h(v_{h,i}) \rightharpoonup \nabla v_i$ weakly in $L^2(\Omega)^d$ for all $i \in \{1,\dots,d\}$. Owing to the uniqueness of the limit in the distribution sense, it is inferred that $\nabla\cdot v = p$. Consider now the terms T_2 and T_3. It is clear that $T_3 \to \int_\Omega f\cdot v$. Furthermore,

$$T_2 = \nu a_h(u_{h,i},v_{h,i}) = \nu \int_\Omega G_h(u_{h,i})\cdot G_h(v_{h,i}) + \nu j_h(u_{h,i},v_{h,i}) = T_{2,1} + T_{2,2}.$$

Owing to the strong convergence of $\{G_h(u_{h,i})\}_{h\in\mathcal{H}}$ in $L^2(\Omega)^d$ and to the weak convergence of $\{G_h(v_{h,i})\}_{h\in\mathcal{H}}$ in $L^2(\Omega)^d$, it is inferred that $T_{2,1} \to \nu \int_\Omega \partial_j u_i \partial_j v_i$. Moreover,

$$|T_{2,2}| \leq c_{\nu,k,\mathcal{P}} |u_h|_{\mathrm{J},\mathcal{F}_h,-1} |v_h|_{\mathrm{J},\mathcal{F}_h,-1} \leq C |u_h|_{\mathrm{J},\mathcal{F}_h,-1},$$

which converges to zero. Collecting the above estimates leads to

$$\limsup \|p_h\|^2_{L^2(\Omega)} \leq \nu \int_\Omega \partial_j u_i \partial_j v_i - \int_\Omega f_i v_i = \int_\Omega p \partial_j v_j = \|p\|^2_{L^2(\Omega)},$$

classically yielding the strong convergence of the pressure in $L^2(\Omega)$. □

Remark 4.1. If the exact solution (u, p) turns out to be more regular and belongs to the broken Sobolev space $H^{k+1}(\mathcal{T}_h)^d \times H^k(\mathcal{T}_h)$, one optimal a priori error estimates of the form $\|(u-u_h, p-p_h)\|_{\mathrm{S}} \leq c_{u,p}\mathrm{size}(\mathcal{T}_h)^k$ can be established; see e.g. [10, 13, 18].

5. The steady incompressible Navier–Stokes equations

In this section the space dimension is either 2 or 3. Let $f \in L^r(\Omega)^d$ with $r = \frac{6}{5}$ if $d = 3$ and $r > 1$ if $d = 2$. Let $\nu > 0$. Consider the steady incompressible Navier–Stokes equations in conservative form

$$\begin{cases} -\nu\Delta u_i + \partial_j(u_i u_j) + \partial_i p = f_i, & \text{in } \Omega, \quad i \in \{1, \dots, d\}, \\ \partial_i u_i = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \int_\Omega p = 0. \end{cases} \tag{59}$$

The weak formulation of this system consists of finding $(u, p) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$ s.t. for all $(v, q) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$,

$$\nu \int_\Omega \partial_j u_i \partial_j v_i + \int_\Omega \partial_j(u_i u_j) v_i - \int_\Omega p \partial_i v_i + \int_\Omega q \partial_i u_i = \int_\Omega f_i v_i. \tag{60}$$

The existence of a weak solution in the above sense, in two and three space dimensions, is a classical result; see, e.g., [32, 24]. The uniqueness of the solution holds only under small data assumptions; see Remark 5.1 below.

5.1. **Design of the convective trilinear form.** We choose the same discrete spaces for the velocity and for the pressure as for the Stokes equations. To allow for some generality in the treatment of the convective term, we introduce two parameters $\alpha_1, \alpha_2 \in \{0, 1\}$ and rewrite the momentum equation in the Navier–Stokes equations as

$$-\nu\Delta u_i + \partial_j(u_i u_j) - \alpha_1 \tfrac{1}{2}(\partial_j u_j) u_i + \alpha_2 \tfrac{1}{2} \partial_i(u_j u_j) + \partial_i \overline{p} = f_i, \tag{61}$$

with the modified pressure

$$\overline{p} \stackrel{\text{def}}{=} p - \alpha_2 \tfrac{1}{2}(u_j u_j). \tag{62}$$

The choice $(\alpha_1, \alpha_2) = (1, 0)$ corresponds to Temam's device (see e.g. [32]) to achieve stability. The choice $(\alpha_1, \alpha_2) = (0, 1)$ has been hinted to in [9]; the modified pressure $\overline{p}$ differs from the Bernoulli pressure but the advantage is that the left-hand side of (61) is in divergence form, thereby lending itself to a conservative discretization. Define on $[H^1_0(\Omega)^d]^3$ the trilinear form

$$t(w, u, v) \stackrel{\text{def}}{=} \int_\Omega \partial_j(w_i u_j) v_i - \alpha_1 \tfrac{1}{2} \int_\Omega (\partial_j w_j) u_i v_i + \alpha_2 \tfrac{1}{2} \int_\Omega \partial_i(w_j u_j) v_i. \tag{63}$$

The discrete counterpart of the trilinear form t is a trilinear form t_h defined on $[U_h]^3$ and for which the following design conditions are relevant.

(T1) For all $v_h \in U_h$,

$$t_h(v_h, v_h, v_h) = 0.$$

(T2) There is c_t, depending on k and $\mathcal{P}$, such that for all $(w_h, u_h, v_h) \in [U_h]^3$,

$$t_h(w_h, u_h, v_h) \le c_t \|w_h\|_{\mathrm{DG}} \|u_h\|_{\mathrm{DG}} \|v_h\|_{\mathrm{DG}}.$$

(T3) Let $\{u_h\}_{h\in\mathcal{H}}$ be a sequence in U_h, bounded in the $\|\cdot\|_{\mathrm{DG}}$-norm and such that there is $u \in H_0^1(\Omega)^d$ s.t. $u_h \to u$ strongly in $L^2(\Omega)^d$ and, for all $i \in \{1, \dots, d\}$, $G_h(u_{h,i}) \rightharpoonup \nabla u_i$ weakly in $L^2(\Omega)^d$. Then, for all $\varphi \in C_c^\infty(\Omega)^d$, as $\mathrm{size}(\mathcal{T}_h) \to 0$,

$$t_h(u_h, u_h, \pi_h \varphi) \to t(u, u, \varphi).$$

(T4) Assume furthermore that, for all $i \in \{1, \dots, d\}$, $G_h(u_{h,i}) \to \nabla u_i$ strongly in $L^2(\Omega)^d$ and that $|u_h|_{\mathrm{J},\mathcal{F}_h,-1} \to 0$. Let $\{v_h\}_{h\in\mathcal{H}}$ be another sequence in U_h, bounded in the $\|\cdot\|_{\mathrm{DG}}$-norm and such that there is $v \in\in H_0^1(\Omega)^d$ s.t. $v_h \to v$ strongly in $L^2(\Omega)^d$. Then, as $\mathrm{size}(\mathcal{T}_h) \to 0$,

$$t_h(u_h, u_h, v_h) \to t(u, u, v).$$

5.2. **Discrete well–posedness and basic stability estimates.** The discrete problem consists of seeking $(u_h, p_h) \in X_h$ s.t.

$$l_h((u_h, p_h), (v_h, q_h)) + t_h(u_h, u_h, v_h) = \int_\Omega f_i v_{h,i}, \qquad \forall (v_h, q_h) \in X_h, \tag{64}$$

where the bilinear form l_h associated with the Stokes equations is defined by (49). In this section, the discrete trilinear form t_h is assumed to satisfy (T1)–(T2) only.

Lemma 5.1 (A priori estimates)**.** *Let $(u_h, p_h) \in X_h$ and assume that (u_h, p_h) solves* (64)*. Then, the following a priori estimates hold:*

$$(\nu\alpha)^2 \|u_h\|_{\mathrm{DG}}^2 + 2\alpha\nu |p_h|_{\mathrm{J},\mathcal{F}_h^i,1}^2 \le \sigma_{r'}^2 \|f\|_{L^r(\Omega)^d}^2, \tag{65}$$

$$c_l \|(u_h, p_h)\|_{\mathrm{S}} \le \sigma_{r'} \|f\|_{L^r(\Omega)^d} + c_t (\nu\alpha)^{-2} (\sigma_{r'} \|f\|_{L^r(\Omega)^d})^2. \tag{66}$$

Proof. To prove (65), simply test (64) with (u_h, p_h), observe that $t_h(u_h, u_h, u_h) = 0$ owing to (T1) and use Lemma 4.2 for the linear part yielding

$$\nu\alpha \|u_h\|_{\mathrm{DG}}^2 + |p_h|_{\mathrm{J},\mathcal{F}_h^i,1}^2 \le \int_\Omega f_i u_{h,i} \le \sigma_{r'} \|f\|_{L^r(\Omega)^d} \|u_h\|_{\mathrm{DG}},$$

whence (65) is easily deduced. To prove (66), use the inf-sup condition in Lemma 4.3 and assumption (T2) to infer

$$c_l \|(u_h, p_h)\|_{\mathrm{S}} \le \sigma_{r'} \|f\|_{L^r(\Omega)^d} + c_t \|u_h\|_{\mathrm{DG}}^2,$$

and conclude using (65). □

To prove the existence of a discrete solution, we use a topological degree argument; see, e.g., [19, 23] for the use of this argument in the convergence analysis of FV schemes and [12] for a general presentation.

Lemma 5.2. *Let V be a finite dimensional functional space equipped with a norm $\|\cdot\|_V$, let $\mu > 0$, and let $\Psi : V \times [0, 1] \to V$ satisfying the following assumptions:*

(i) *Ψ is continuous;*
(ii) *$\Psi(\cdot, 0)$ is an affine function and the equation $\Psi(v, 0) = 0$ has a solution $v \in V$ such that $\|v\|_V < \mu$;*
(iii) *For any $(v, \rho) \in V \times [0, 1]$, $\Psi(v, \rho) = 0$ implies $\|v\|_V \ne \mu$.*

Then, there exists $v \in V$ such that $\Psi(v, 1) = 0$ and $\|v\|_V < \mu$.

Proposition 5.1. *For all $h \in \mathcal{H}$, the discrete problem* (64) *admits at least one solution $(u_h, p_h) \in X_h$.*

Proof. To apply Lemma 5.2, let $V = X_h$ and define the mapping $\Psi : X_h \times [0,1] \to X_h$ such that for (u_h, p_h) given in X_h and ρ given in $[0,1]$, $(\xi_h, \zeta_h) \stackrel{\text{def}}{=} \Psi((u_h, p_h), \rho) \in X_h$ is defined such that for all $(v_h, q_h) \in X_h$,

$$(\xi_h, v_h)_{L^2(\Omega)^d} = l_h((u_h, p_h), (v_h, 0)) + \rho t_h(u_h, u_h, v_h) - \int_\Omega f_i v_{h,i},$$
$$(\zeta_h, q_h)_{L^2(\Omega)} = l_h((u_h, p_h), (0, q_h)).$$

Observing that l_h is continuous on $X_h \times X_h$ for the $\|\cdot\|_{\mathrm{S}}$-norm, using (T2) and the equivalence of norms in finite dimension, it is inferred that Ψ is continuous. Furthermore, point (ii) in Lemma 5.2 results from the a priori estimate for the Stokes equations. In addition, because of (T1), if $(u_h, p_h) \in X_h$ is such that $\Psi((u_h, p_h), \rho) = 0$ for some $\rho \in [0,1]$, then (u_h, p_h) is bounded independently of ρ. This concludes the proof. □

5.3. **Convergence analysis.** In this section, we are now interested in the convergence of a sequence $\{(u_h, p_h)\}_{h\in\mathcal{H}}$ of solutions to the discrete problem (64) towards a solution (u, p) of the Navier–Stokes equations (60). The same convergence result can be established as for the Stokes equations. The only difference is that, because we do not make a smallness assumption on the data, there is no uniqueness result available at the continuous level, and thus only the convergence of subsequences (and not of the whole sequence) is obtained.

Theorem 5.1 (Convergence for Navier–Stokes equations)**.** *Let $\{(u_h, p_h)\}_{h\in\mathcal{H}}$ be a sequence of approximate solutions generated by solving the discrete problems* (64) *on the admissible meshes $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Assume* (T1)–(T3)*. Then, as* $\text{size}(\mathcal{T}_h) \to 0$, *up to a subsequence,*

$$u_h \to u, \quad \textit{in } L^2(\Omega)^d, \tag{67}$$
$$\nabla_h u_h \to \nabla u, \quad \textit{in } L^2(\Omega)^{d,d}, \tag{68}$$
$$|u_h|_{\mathrm{J},\mathcal{F}_h,-1} \to 0, \tag{69}$$
$$p_h \rightharpoonup \overline{p}, \quad \textit{weakly in } L^2(\Omega), \tag{70}$$
$$|p_h|_{\mathrm{J},\mathcal{F}_h^i,1} \to 0, \tag{71}$$

where $(u, \overline{p} + \alpha_2 \frac{1}{2}(u_j u_j)) \in H_0^1(\Omega) \times L_0^2(\Omega)$ is a solution to (60)*. Moreover, if* (T4) *also holds, then $p_h \to \overline{p}$ in $L^2(\Omega)$.*

Proof. (i) Proceeding as for the Stokes equations, it is clear that there is $(u, \overline{p}) \in H_0^1(\Omega) \times L_0^2(\Omega)$ s.t., up to a subsequence, $u_h \to u$ strongly in $L^2(\Omega)^d$, $G_h(u_{h,i}) \rightharpoonup \nabla u_i$ weakly in $L^2(\Omega)^d$ for all $i \in \{1, \ldots, d\}$ and $p_h \rightharpoonup \overline{p}$ weakly in $L^2(\Omega)$.
(ii) Identification of the limit. Using (T3) and proceeding as for the Stokes equations to treat the linear part, it is inferred that for all $\varphi \in C_c^\infty(\Omega)^d$,

$$\nu \int_\Omega \partial_j u_i \partial_j \varphi_i + t(u, u, \varphi) - \int_\Omega \overline{p} \partial_j \varphi_j = \int_\Omega f_i \varphi_i.$$

and that for all $\psi \in C_c^\infty(\Omega)/\mathbb{R}$,

$$\int_\Omega \psi \partial_j u_j = 0.$$

Hence, $(u, \overline{p} + \alpha_2 \frac{1}{2}(u_j u_j))$ solves the incompressible Navier–Stokes equations.
(iii) Strong convergence of the velocity and of the jumps. Proceeding as for the Stokes equations, (T1) yields the strong convergence of the piecewise velocity gradient in $L^2(\Omega)^d$ and the convergence to zero of the jump seminorms $|u_h|_{\mathrm{J},\mathcal{F}_h,-1}$ and $|p_h|_{\mathrm{J},\mathcal{F}_h^i,1}$.
(iv) Strong convergence of the pressure. Proceeding as for the Stokes equations yields

$$\begin{aligned}\|p_h\|_{L^2(\Omega)}^2 &\leq c_{\Omega,k,\mathcal{P}} |p_h|_{\mathrm{J},\mathcal{F}_h^i,1} \|p_h\|_{L^2(\Omega)} + \nu a_h(u_{h,i}, v_{h,i}) + t_h(u_h, u_h, v_h) - \int_\Omega f_i v_{h,i} \\ &= T_1 + T_2 + T_3 - T_4.\end{aligned}$$

The convergence of T_1, T_2 and T_4 is treated as for the Stokes equations. Furthermore, the convergence of T_3 results from assumption (T4). As a result,

$$\begin{aligned}\limsup \|p_h\|_{L^2(\Omega)}^2 &\leq \nu \int_\Omega \partial_j u_i \partial_j v_i + t(u,u,v) - \int_\Omega f_i v_i \\ &= \int_\Omega p(\partial_i v_i) + \alpha_2 \tfrac{1}{2} \int_\Omega \partial_i (u_j u_j) v_i \\ &= \int_\Omega \overline{p}(\partial_i v_i) = \|\overline{p}\|_{L^2(\Omega)}^2,\end{aligned}$$

concluding the proof. □

Remark 5.1. Under a smallness condition of the form

$$c_{\Omega,k,\mathcal{P}} \nu^{-2} \|f\|_{L^r(\Omega)^d} < 1,$$

uniqueness of the weak solution of (60) classically holds, so that the conclusions (67)–(71) of Theorem 5.1 apply to the whole sequence $\{(u_h, p_h)\}_{h\in\mathcal{H}}$. Moreover, the convergence of the fixed-point iterative scheme

$$l_h((u_h^{k+1}, p_h^{k+1}), (v_h, q_h)) + t_h(u_h^k, u_h^{k+1}, v_h) = \int_\Omega f_i v_{h,i}, \qquad \forall (v_h, q_h) \in X_h,$$

can be proven using standard arguments.

5.4. **Examples.** Define for $(w_h, u_h, v_h) \in [U_h]^3$,

$$\begin{aligned}t_h(w_h, u_h, v_h) = &\int_\Omega (w_h \cdot \nabla_h u_h) \cdot v_h - \sum_{F\in\mathcal{F}_h^i} \int_F \{\!\{w_h\}\!\} \cdot \nu_F [\![u_h]\!] \cdot \{\!\{v_h\}\!\} \\ &+ \int_\Omega \tfrac{1}{2} (\nabla_h \cdot w_h)(u_h \cdot v_h) - \sum_{F\in\mathcal{F}_h} \int_F [\![w_h]\!] \cdot \nu_F \tfrac{1}{2} \{\!\{u_h \cdot v_h\}\!\}.\end{aligned} \tag{72}$$

This choice corresponds to $(\alpha_1, \alpha_2) = (1, 0)$. The resulting DG method is not conservative, but contains a source term proportional to the divergence of the discrete velocity (still converging to zero as the mesh is refined).

Proposition 5.2. *Let t_h be defined by* (72)*. Then, assumptions* (T1)–(T4) *hold.*

Proof. The verification of (T1) is straightforward. Assumption (T2) results from the Sobolev embedding with $q = 4$ and trace inequalities. To prove (T3) and (T4), observe first that for all $v_h \in U_h$,

$$t_h(u_h,u_h,v_h) = \int_\Omega u_h\cdot\mathcal{G}_h^{2k}(u_{h,i})v_{h,i} + \frac{1}{4}\sum_{F\in\mathcal{F}_h^i}\int_F [\![u_{h,i}]\!]\nu_F\cdot[\![u_h]\!][\![v_{h,i}]\!] + \int_\Omega D_h^{2k}(u_h)\tfrac{1}{2}u_{h,i}v_{h,i} = T_1+T_2+T_3.$$

To prove (T3), take $v_h = \pi_h\varphi$ with $\varphi \in C_c^\infty(\Omega)^d$. Owing to the discrete Sobolev embedding with $q=4$, the sequences $\{u_h\}_{h\in\mathcal{H}}$ and $\{\pi_h\varphi\}_{h\in\mathcal{H}}$ are bounded in $L^4(\Omega)^d$. Hence, Lebesgue's Dominated convergence Theorem implies that, up to a subsequence, $u_h\pi_h\varphi_i$ converges to $u\varphi_i$ in $L^2(\Omega)^d$. In addition, $\{\mathcal{G}_h^{2k}(u_{h,i})\}_{h\in\mathcal{H}}$ weakly converges to ∇u_i in $L^2(\Omega)^d$. As a result, T_1 converges to $\int_\Omega u_j(\partial_j u_i)\varphi_i$. Similarly, T_3 converges to $\int_\Omega \frac{1}{2}(\partial_j u_j)u_i\varphi_i$. Furthermore, $T_2 \to 0$ since $|u_h|_{\mathrm{J},\mathcal{F}_h,-1}$ is bounded and $\|[\![\pi_h\varphi_i]\!]\|_{L^\infty(F)}$ converges to zero. Therefore, as size$(\mathcal{T}_h)\to 0$,

$$t_h(u_h,u_h,\pi_h\varphi) \to \int_\Omega u_j(\partial_j u_i)\varphi_i + \int_\Omega \tfrac{1}{2}(\partial_j u_j)u_i\varphi_i = \int_\Omega [\partial_j(u_iu_j) - \tfrac{1}{2}(\partial_j u_j)u_i]\varphi_i.$$

Assumption (T4) is proven similarly for the terms T_1 and T_3. To prove that T_2 converges to zero, observe that $|u_h|_{\mathrm{J},\mathcal{F}_h,-1}$ converges to zero and that $\|[\![v_h]\!]\|_{L^\infty(F)} \le c_{k,\mathcal{P}}h_F^{-1}$ owing to a trace inequality. This concludes the proof. □

Define now for $(w_h,u_h,v_h)\in[U_h]^3$,

$$(73)\quad \begin{aligned} t_h(w_h,u_h,v_h) = &-\int_\Omega (w_{h,i}u_h\cdot\nabla_h v_{h,i} + \sum_{F\in\mathcal{F}_h^i}\int_F \nu_F\cdot\{\!\{u_h\}\!\}\{\!\{w_{h,i}\}\!\}[\![v_{h,i}]\!] \\ &+ \int_\Omega \tfrac{1}{2}v_h\cdot\nabla_h(u_{h,i}w_{h,i}) - \sum_{F\in\mathcal{F}_h^i}\int_F \nu_F\cdot\{\!\{v_h\}\!\}\tfrac{1}{2}[\![u_{h,i}w_{h,i}]\!]. \end{aligned}$$

This choice corresponds to $(\alpha_1,\alpha_2)=(0,1)$. The salient feature of the resulting DG method is that it is locally conservative.

Proposition 5.3. *Let t_h be defined by* (73). *Then, assumptions* (T1)–(T4) *hold.*

Proof. Assumptions (T1)–(T2) can be readily verified. To prove (T3) and (T4), observe that for all $v_h\in U_h$,

$$t_h(u_h,u_h,\pi_h\varphi) = -\int_\Omega u_{h,i}u_h\cdot\mathcal{G}_h^{2k}(v_{h,i}) - \frac{1}{4}\sum_{F\in\mathcal{F}_h^i}\int_F \nu_F\cdot[\![u_h]\!][\![u_{h,i}]\!][\![v_{h,i}]\!] - \int_\Omega \tfrac{1}{2}u_{h,i}u_{h,i}\mathcal{D}_h^{2k}(v_h),$$

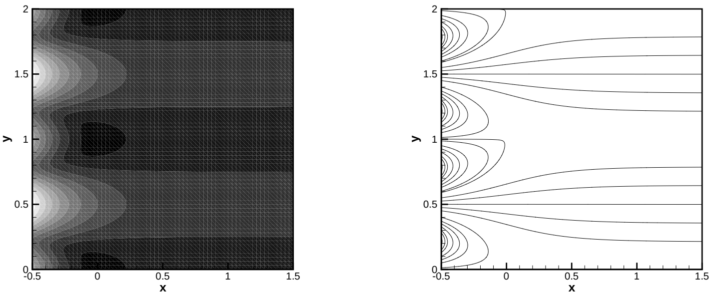
where $\mathcal{D}_h^{2k}(v_h) \stackrel{\text{def}}{=} \mathcal{G}_h^{2k}(v_{h,i})\cdot e_i$. Then, proceed as in the previous proof to infer that for all $\varphi\in C_c^\infty(\Omega)^d$, as size$(\mathcal{T}_h)\to 0$,

$$t_h(u_h,u_h,\pi_h\varphi) \to \int_\Omega [\partial_j(u_iu_j) + \tfrac{1}{2}\partial_i(u_ju_j)]\varphi_i,$$

along with a similar result for (T4). □

TABLE 1. Convergence results for the trilinear form defined by (72). We have set $e_h = (e_{h,u}, e_{h,p}) \stackrel{\text{def}}{=} (u - u_h, p - p_h)$.

mesh	h	$\lVert e_{h,u}\rVert_{L^2(\Omega)^d}$	order	$\lVert e_{h,p}\rVert_{L^2(\Omega)}$	order	$\lVert e_h\rVert_{\mathrm{S}}$	order
1	$5.00e-1$	$8.87e-01$	–	$1.62e+00$	–	$1.19e+01$	–
2	$2.50e-1$	$2.39e-01$	1.89	$6.11e-01$	1.41	$7.26e+00$	0.71
3	$1.25e-1$	$5.94e-02$	2.01	$2.01e-01$	1.60	$3.68e+00$	0.98
4	$6.25e-2$	$1.59e-02$	1.90	$7.40e-02$	1.44	$1.85e+00$	0.99
5	$3.12e-2$	$4.17e-03$	1.93	$3.14e-02$	1.23	$9.25e-01$	1.00



(a) Velocity module (b) Stream lines

FIGURE 2. Plot of Kovasznay's solution for $k = 1$ and mesh 5.

Remark 5.2. Upwinding can be introduced in the discrete trilinear forms t_h defined by (72) or (73) by adding a term of the form

$$\sum_{F\in\mathcal{F}_h^i} \theta_F \int_F |\{\!\{w_h\}\!\}\cdot\nu_F| [\![u_h]\!]\cdot[\![v_h]\!],$$

and replacing the design assumption (T1) by the requirement that t_h be nonnegative, which is sufficient to derive all the necessary a priori estimates and the convergence result of Theorem 5.1. Here, the parameter $\theta_F \in [0,1]$ depends on the local Péclet number.

5.5. **Numerical experiment.** To verify the asymptotic convergence properties of the method defined by (72), we have considered the analytical solution proposed in [27] on the square domain $\Omega \stackrel{\text{def}}{=} (-0.5, 1.5) \times (0, 2)$,

$$u_1 = 1 - e^{-\pi x_2}\cos(2\pi x_2), \quad u_2 = -\frac{1}{2}e^{\pi x_1}\sin(2\pi x_2), \quad p = -\frac{1}{2}e^{\pi x_1}\cos(2\pi x_2) - \widetilde{p},$$

where $\widetilde{p} \stackrel{\text{def}}{=} \frac{1}{\text{meas}(\Omega)}\int_\Omega -\frac{1}{2}e^{\pi x_1}\cos(2\pi x_2) \simeq -0.920735694$ ensures zero-mean for the pressure, $\nu = \frac{1}{3\pi}$ and $f = 0$. The example was run on a family of uniformly refined triangular meshes with mesh sizes ranging from 0.5 down to 0.03125, labeled with progressive numbers from 1 to 5 in Table 1. The nonlinear problem was solved by the exact Newton algorithm with tolerance set to 10^{-6}; the linear systems were solved using the direct solver available in PETSc. According to Table 1, the method converges with optimal order in the energy norm defined by (51). The method defined by (73) was also tested, and the corresponding asymptotic convergence rates were observed to be suboptimal by half an order. Further tests are out of the scope of the present paper and will receive extensive attention in a future work.

6. Discrete functional analysis in DG spaces

Let $1 \le p < +\infty$ and let $k \ge 1$ be an integer. Equip the DG finite element space V_h^k defined by (4) with the norm

$$\|v_h\|_{\mathrm{DG},p}^p \stackrel{\mathrm{def}}{=} \sum_{T\in\mathcal{T}_h} \int_T |\nabla v_h|_{\ell^p}^p + \sum_{F\in\mathcal{F}_h} \frac{1}{h_F^{p-1}} \int_F |[\![v_h]\!]|^p, \tag{74}$$

where $|\cdot|_{\ell^p}$ denotes the ℓ^p-norm in $\mathbb{R}^d$ so that $|\nabla v_h|_{\ell^p}^p = \sum_{i=1}^d |\partial_i v_h|^p$. Recall that Ω is a open bounded connected subset of $\mathbb{R}^d$ $(d > 1)$ whose boundary is a finite union of parts of hyperplanes. In this section, the mesh family $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ used to build the DG spaces is assumed to satisfy only assumptions (i)–(iv) in Definition 2.1.

The material contained in this section, which is closely inspired from that derived in [21] for discrete spaces of piecewise constant functions, deals with the extension to DG spaces of two key results of functional analysis, namely Sobolev embeddings and compactness criteria in $L^p(\Omega)$. These results are presented here in a non-Hilbertian setting which is more general than that needed to analyze the Navier–Stokes equations. We have made this choice because the results below are of independent interest to analyze other nonlinear problems. We also observe that we deal here with functional analysis in DG spaces and not in broken Sobolev spaces.

Lemma 6.1. *For all $1 \le s < t < +\infty$, the following holds for all $v_h \in V_h^k$,*

$$\|v_h\|_{\mathrm{DG},s} \le c_{d,\varrho_1,|\Omega|,s,t} \|v_h\|_{\mathrm{DG},t}. \tag{75}$$

Proof. Observing that for all $x \in \mathbb{R}^d$, $|x|_{\ell^s} \le d^{\frac{1}{s}-\frac{1}{t}} |x|_{\ell^t}$ and using Hölder's inequality with $\pi = \frac{t}{s} > 1$ and $\pi' = \frac{\pi}{\pi-1}$ yields

$$\begin{aligned}
\|v_h\|_{\mathrm{DG},s}^s &= \sum_{T\in\mathcal{T}_h} \int_T |\nabla v_h|_{\ell^s}^s + \sum_{F\in\mathcal{F}_h} \frac{1}{h_F^{s-1}} \int_F |[\![v_h]\!]|^s \\
&\le \sum_{T\in\mathcal{T}_h} \int_T d^{\frac{1}{\pi'}} |\nabla v_h|_{\ell^t}^s + \sum_{F\in\mathcal{F}_h} \int_F h_F^{\frac{1}{\pi'}} h_F^{\frac{1}{\pi}(1-t)} |[\![v_h]\!]|^s \\
&\le \left(\sum_{T\in\mathcal{T}_h} d \int_T 1^{\pi'} \right)^{\frac{1}{\pi'}} \left(\sum_{T\in\mathcal{T}_h} \int_T |\nabla v_h|_{\ell^t}^t \right)^{\frac{1}{\pi}} \\
&\quad + \left(\sum_{F\in\mathcal{F}_h} h_F \int_F 1^{\pi'} \right)^{\frac{1}{\pi'}} \left(\sum_{F\in\mathcal{F}_h} \frac{1}{h_F^{t-1}} \int_F |[\![v_h]\!]|^t \right)^{\frac{1}{\pi}} \\
&\le ((d+\varrho_1)|\Omega|)^{\frac{1}{\pi'}} \|v_h\|_{\mathrm{DG},t}^s,
\end{aligned}$$

using (1), whence the conclusion follows. □

Lemma 6.2. *For $v \in L^1(\mathbb{R}^d)$, define*

$$\|v\|_{\mathrm{BV}} = \sum_{i=1}^d \sup\{\textstyle\int_{\mathbb{R}^d} u\partial_i\varphi;\ \varphi \in C_c^\infty(\mathbb{R}^d),\ \|\varphi\|_{L^\infty(\mathbb{R}^d)} \le 1\},$$

and set $\mathrm{BV} = \{v \in L^1(\mathbb{R}^d);\ \|v\|_{\mathrm{BV}} < +\infty\}$*. Then, extending discrete functions in* V_h^k *by zero outside* Ω*, there holds* $V_h^k \subset \mathrm{BV}$ *and for all* $1 \le p < +\infty$*,*

$$\forall v_h \in V_h^k, \qquad \|v_h\|_{\mathrm{BV}} \le c_{d,\varrho_1,|\Omega|,p} \|v_h\|_{\mathrm{DG},p}. \tag{76}$$

Proof. Clearly, owing to Lemma 6.1, it suffices to prove (76) for $p = 1$. Integrating by parts, it is clear that for all $v_h \in V_h^k$ and for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\|\varphi\|_{L^\infty(\mathbb{R}^d)} \le 1$,

$$\int_{\mathbb{R}^d} v_h \partial_i \varphi = -\int_{\mathbb{R}^d} (e_i\cdot\nabla_h v_h)\varphi + \sum_{F\in\mathcal{F}_h} \int_F e_i\cdot\nu_F [\![v_h]\!]\varphi \le \|v_h\|_{\mathrm{DG},1}.$$

Hence, $\|v_h\|_{\mathrm{BV}} \le d\|v_h\|_{\mathrm{DG},1}$, completing the proof. □

Remark 6.1. In this section we could have allowed the case $k = 0$, although the derived results are not as interesting as for $k \ge 1$ because $\|\cdot\|_{\mathrm{DG},p}$ is not the natural norm with which to equip the space V_h^0 when working with FV approximations to nonlinear second-order PDE's. Indeed, on V_h^0, the first term on the right-hand side of (74) (the broken gradient) drops out, and this entails that a length scale different from h_F must be used for the jump term, thereby also requiring an additional (mild) assumption on the mesh family; see [21] for the analysis in this case.

Remark 6.2. The observation that the $\|\cdot\|_{\mathrm{DG},2}$-norm controls the BV norm can also be found in [29] in the framework of linear elasticity.

6.1. Discrete Sobolev embeddings.

Theorem 6.1 (Discrete Sobolev embeddings)**.** *For all* q *such that*

(i) $1 \le q \le p^* \stackrel{\mathrm{def}}{=} \frac{pd}{d-p}$ *if* $1 \le p < d$*;*
(ii) $1 \le q < +\infty$ *if* $d \le p < +\infty$*;*

there is $\sigma_{q,p}$ *such that*

$$\forall v_h \in V_h^k, \qquad \|v_h\|_{L^q(\Omega)} \le \sigma_{p,q}\|v_h\|_{\mathrm{DG},p}. \tag{77}$$

The constant $\sigma_{q,p}$ *additionally depends on* k*,* $|\Omega|$*, and* $\mathcal{P}$*. In particular, for the choice* $q = p$ *which is always possible,*

$$\forall v_h \in V_h^k, \qquad \|v_h\|_{L^p(\Omega)} \le \sigma_{p,p}\|v_h\|_{\mathrm{DG},p}. \tag{78}$$

Proof. We follow L. Nirenberg's proof of Sobolev embeddings.
(i) The case $p = 1$. Set $1^* \stackrel{\mathrm{def}}{=} \frac{d}{d-1}$. Then, owing to a classical result (see, e.g. [21] for a proof), for all $v \in \mathrm{BV}$,

$$\|v\|_{L^{1^*}(\mathbb{R}^d)} \le \frac{1}{2d}\|v\|_{\mathrm{BV}}.$$

Extending discrete functions in V_h^k by zero outside Ω, Lemma 6.2 yields

$$\|v_h\|_{L^{1^*}(\mathbb{R}^d)} \le \frac{1}{2}\|v_h\|_{\mathrm{DG},1}, \tag{79}$$

i.e., (77) for $p = 1$ and $q = 1^*$ with $\sigma_{1,1^*} = \frac{1}{2}$, and hence for all $1 \le q \le 1^*$ since Ω is bounded.
(ii) The case $1 < p < d$. Set $\alpha = \frac{p(d-1)}{d-p}$ and observe that $\alpha > 1$. Considering the function $|v_h|^\alpha$ (extended by zero outside Ω) and using (79) yields

$$2\left(\int_\Omega |v_h|^{p^*}\right)^{\frac{d-1}{d}} \le \sum_{T\in\mathcal{T}_h}\int_T |\nabla|v_h|^\alpha|_{\ell^1} + \sum_{F\in\mathcal{F}_h}\int_F |[\![|v_h|^\alpha]\!]| \equiv T_1 + T_2. \tag{80}$$

Observe that a.e. in each $T \in \mathcal{T}_h$, $|\partial_i |v_h|^\alpha| = \alpha |v_h|^{\alpha-1} |\partial_i v_h|$ for all $i \in \{1, \ldots, d\}$ so that $|\nabla |v_h|^\alpha|_{\ell^1} = \alpha |v_h|^{\alpha-1} |\nabla v_h|_{\ell^1}$. Using Hölder's inequality with p and $q = \frac{p}{p-1}$, the first term in (80) is bounded as

$$
\begin{aligned}
|T_1| &\leq \alpha \left(\sum_{T \in \mathcal{T}_h} \int_T |v_h|^{q(\alpha-1)} \right)^{\frac{1}{q}} \left(\sum_{T \in \mathcal{T}_h} \int_T |\nabla v_h|_{\ell^1}^p \right)^{\frac{1}{p}} \\
&\leq \alpha d^{\frac{p-1}{p}} \left(\int_\Omega |v_h|^{p^*} \right)^{\frac{1}{q}} \left(\sum_{T \in \mathcal{T}_h} \int_T |\nabla v_h|_{\ell^p}^p \right)^{\frac{1}{p}}.
\end{aligned}
$$

Furthermore, observing that $|[\![|v_h|^\alpha]\!]| \leq 2\alpha \{\!\{ |v_h|^{\alpha-1} \}\!\} |[\![v_h]\!]|$ and using again Hölder's inequality, it is inferred that the second term in (80) is bounded as

$$
\begin{aligned}
|T_2| &\leq \alpha \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F h_F^{\frac{1}{q}} |v_{h|T}|^{\alpha-1} h_F^{-\frac{1}{q}} |[\![v_h]\!]| \\
&\leq \alpha \left(\sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F h_F |v_{h|T}|^{p^*} \right)^{\frac{1}{q}} \left(\sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \frac{1}{h_F^{p-1}} \int_F |[\![v_h]\!]|^p \right)^{\frac{1}{p}} \\
&\leq \alpha 2^{\frac{1}{p}} \tau_{p^*,k}^{\frac{1}{q}} \left(\int_\Omega |v_h|^{p^*} \right)^{\frac{1}{q}} \left(\sum_{F \in \mathcal{F}_h} \frac{1}{h_F^{p-1}} \int_F |[\![v_h]\!]|^p \right)^{\frac{1}{p}},
\end{aligned}
$$

where for $s \in \mathbb{R}_+$, $\tau_{s,k}$ is the constant in the trace inequality

$$
\forall \zeta \in \mathbb{P}_k(T), \qquad \sum_{F \subset \partial T} h_F \int_F |\zeta|^s \leq \tau_{s,k} \int_T |\zeta|^s,
$$

valid uniformly for all $h \in \mathcal{H}$ and for all $T \in \mathcal{T}_h$. This leads to

$$
\begin{aligned}
2 \left(\int_\Omega |v_h|^{p^*} \right)^{\frac{d-1}{d}} &\leq \alpha (d + 2^{\frac{1}{p-1}} \tau_{p^*,k})^{\frac{1}{q}} \left(\int_\Omega |v_h|^{p^*} \right)^{\frac{1}{q}} \|v_h\|_{\mathrm{DG},p} \\
&\leq \alpha (d^{\frac{1}{q}} + 2^{\frac{1}{p}} \tau_{p^*,k}^{\frac{1}{q}}) \left(\int_\Omega |v_h|^{p^*} \right)^{\frac{1}{q}} \|v_h\|_{\mathrm{DG},p}.
\end{aligned}
$$

Observing that $\frac{d-1}{d} - \frac{1}{q} = \frac{1}{p^*}$ yields (77).

(iii) The case $d \leq p < +\infty$. Fix any q_1 such that $p < q_1 < +\infty$ and set $p_1 = \frac{dq_1}{d+q_1}$ so that $p_1 < d$ and $p_1^* = q_1$. Then, owing to point (ii) in this proof, it is inferred that for all $v_h \in V_h^k$,

$$
\|v_h\|_{L^{q_1}(\Omega)} \leq \sigma_{p_1,q_1} \|v_h\|_{\mathrm{DG},p_1},
$$

and the conclusion follows from Lemma 6.1 since $p_1 \leq p$. □

6.2. **Compactness.** In this section we are interested in sequences $\{v_h\}_{h \in \mathcal{H}}$ in V_h^k which are bounded in the $\|\cdot\|_{\mathrm{DG}}$-norm.

Theorem 6.2 (Compactness)**.** *Let $\{v_h\}_{h \in \mathcal{H}}$ be a sequence in V_h^k and assume that this sequence is bounded in the $\|\cdot\|_{\mathrm{DG},p}$-norm. Then, the family $\{v_h\}_{h \in \mathcal{H}}$ is relatively compact in $L^p(\Omega)$ (and also in $L^p(\mathbb{R}^d)$ taking $v_h = 0$ outside Ω).*

Proof. Extending the functions v_h by zero outside Ω and observing that (see, e.g. [21]) for all $\xi \in \mathbb{R}^d$,

$$
\|v_h(\cdot + \xi) - v_h\|_{L^1(\mathbb{R}^d)} \leq |\xi|_{\ell^1} \|v_h\|_{\mathrm{BV}} \leq C |\xi|_{\ell^1},
$$

because of the boundedness of the sequence $\{v_h\}_{h\in\mathcal{H}}$ in the $\|\cdot\|_{\mathrm{DG},p}$-norm (and hence in the BV-norm owing to Lemma 6.2), Kolmogorov's Compactness Criterion yields that the family $\{v_h\}_{h\in\mathcal{H}}$ is relatively compact in $L^1(\mathbb{R}^d)$. Owing to the Sobolev embedding (78), this sequence is also bounded in $L^p(\mathbb{R}^d)$; hence, it is also relatively compact in $L^p(\mathbb{R}^d)$. Finally, the relative compactness also holds in $L^p(\Omega)$ since the functions v_h have been extended by zero outside Ω. □

Theorem 6.3 (Regularity of the limit)**.** *Let* $1 < p < +\infty$*. Let* $\{v_h\}_{h\in\mathcal{H}}$ *be a sequence in* V_h^k *and assume that this sequence is bounded in the* $\|\cdot\|_{\mathrm{DG},p}$*-norm. Assume that* $\mathrm{size}(\mathcal{T}_h) \to 0$*. Then, there exists* $v \in W_0^{1,p}(\Omega)$ *such that, up to a subsequence,* $v_h \to v$ *in* $L^p(\Omega)$*.*

Proof. Owing to Theorem 6.2, there is $v \in L^p(\Omega)$ such that, up to a subsequence, $\{v_h\}_{h\in\mathcal{H}}$ converges to v in $L^p(\Omega)$. It remains to prove that $v \in W_0^{1,p}(\Omega)$. To this purpose, we again extend the functions v_h by zero outside Ω and we construct a discrete gradient converging, at least in the distribution sense over $\mathbb{R}^d$, to ∇v.
(1) Consider the lifting operators r_F^0 and R_h^0 defined in §2.3 and recall that the support of r_F^0 consists of the one or two mesh elements of which F is a face. Hence,

$$\begin{aligned}\|R_h^0([\![v_h]\!])\|_{L^p(\Omega)^d}^p &= \sum_{T\in\mathcal{T}_h}\int_T \left|\sum_{F\subset\partial T} r_F^0([\![v_h]\!])\right|_{\ell^p}^p \\ &\le \sum_{T\in\mathcal{T}_h}\int_T N_\partial^{p-1}\sum_{F\subset\partial T}|r_F^0([\![v_h]\!])|_{\ell^p}^p = N_\partial^{p-1}\sum_{F\in\mathcal{F}_h}\|r_F^0([\![v_h]\!])\|_{L^p(\Omega)^d}^p.\end{aligned}$$

Furthermore, setting for all $i \in \{1,\dots,d\}$, $y_{h,i} = |r_{F,i}^0([\![v_h]\!])|^{p-2} r_{F,i}^0([\![v_h]\!])$, observing that $y_h \in [V_h^0]^d$ and using Hölder's inequality with p and $q = \frac{p}{p-1}$ yields

$$\begin{aligned}\|r_F^0([\![v_h]\!])\|_{L^p(\Omega)^d}^p &= \int_\Omega y_h\cdot r_F^0([\![v_h]\!]) = \int_F \{\!\{y_h\}\!\}\cdot\nu_F[\![v_h]\!] \\ &\le 2^{-\frac{1}{q}}\left(\sum_{T;F\subset\partial T} h_F\int_F |y_{h|T}\cdot\nu_F|^q\right)^{\frac{1}{q}}\left(\frac{1}{h_F^{p-1}}\int_F |[\![v_h]\!]|^p\right)^{\frac{1}{p}} \\ &\le 2^{-\frac{1}{q}}\left(\sum_{T;F\subset\partial T} h_F d^{\frac{q}{p}}\int_F |r_F^0([\![v_h]\!])|_{\ell^p}^p\right)^{\frac{1}{q}}\left(\frac{1}{h_F^{p-1}}\int_F |[\![v_h]\!]|^p\right)^{\frac{1}{p}} \\ &\le c_{d,p,k,\mathcal{P}}\|r_F^0([\![v_h]\!])\|_{L^p(\Omega)^d}^{\frac{p}{q}}\left(\frac{1}{h_F^{p-1}}\int_F |[\![v_h]\!]|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Collecting the above bounds yields

$$\|R_h^0([\![v_h]\!])\|_{L^p(\Omega)^d} \le c_{d,p,k,\mathcal{P}}\left(\sum_{F\in\mathcal{F}_h}\frac{1}{h_F^{p-1}}\int_F |[\![v_h]\!]|^p\right)^{\frac{1}{p}}.$$

Then, upon defining the approximate gradient $G_h^0(v_h) = \nabla_h v_h - R_h^0([\![v_h]\!]) \in [V_h^k]^d$ and extending it by zero outside Ω, it is inferred that $\|G_h^0(v_h)\|_{L^p(\mathbb{R}^d)^d} \le c_{d,p,k,\mathcal{P}}\|v_h\|_{\mathrm{DG},p}$. Hence, the sequence $\{G_h^0(v_h)\}_{h\in\mathcal{H}}$ is bounded in $L^p(\mathbb{R}^d)^d$, and thus since $p > 1$, up to a subsequence, $G_h^0(v_h) \rightharpoonup w$ weakly in $L^p(\mathbb{R}^d)^d$.

(ii) Let $\varphi \in C_c^\infty(\mathbb{R}^d)^d$ and observe that

$$\int_{\mathbb{R}^d} G_h^0(v_h)\cdot\varphi = -\int_{\mathbb{R}^d} v_h(\nabla\cdot\varphi) - \int_{\mathbb{R}^d} R_h^0([\![v_h]\!])\cdot(\varphi - \pi_h^0\varphi) + \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\varphi - \pi_h^0\varphi\}\!\}\cdot\nu_F[\![v_h]\!]$$
$$= T_1 + T_2 + T_3.$$

Letting $\text{size}(\mathcal{T}_h) \to 0$, we observe that $T_1 \to -\int_{\mathbb{R}^d} v(\nabla\cdot\varphi)$ and that $T_2 \to 0$ since $\|\varphi-\pi_h^0\varphi\|_{L^\infty(\mathbb{R}^d)^d} \to 0$ and $\|R_h^0([\![v_h]\!])\|_{L^p(\mathbb{R}^d)^d}$ is bounded. Furthermore, proceeding as usual with $q = \frac{p}{p-1}$ yields

$$T_3 \le c_{\mathcal{P}}\|\varphi - \pi_h^0\varphi\|_{L^\infty(\mathbb{R}^d)^d}|\Omega|^{\frac{1}{q}}\left(\sum_{F\in\mathcal{F}_h}\frac{1}{h_F^{p-1}}\int_F|[\![v_h]\!]|^p\right)^{\frac{1}{p}} \le C\|\varphi - \pi_h^0\varphi\|_{L^\infty(\mathbb{R}^d)^d}$$

whence it is inferred that $T_3 \to 0$. As a result,

$$\int_{\mathbb{R}^d} w\cdot\varphi = \lim_{\text{size}(\mathcal{T}_h)\to 0}\int_{\mathbb{R}^d} G_h^0(v_h)\cdot\varphi = -\int_{\mathbb{R}^d} v(\nabla\cdot\varphi).$$

Hence, $w = \nabla v$ so that $v \in W^{1,p}(\Omega)$, and since v is zero outside Ω, $v \in W_0^{1,p}(\Omega)$. □

Remark 6.3. For $p = 2$, lifting operators using a higher polynomial degree $l \ge 1$ can also be considered as in the proof of Theorem 2.2. The difficulty for $p \ne 2$ is that the vector y_h in the above proof is not necessarily polynomial-valued.

Acknowledgment. The authors are thankful to R. Eymard (University Paris-Est) for fruitful discussions.

References

[1] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19:742–760, 1982.

[2] D. N. Arnold, F. Brezzi, B. Cockburn, and D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2002.

[3] F. Bassi, A. Crivellini, D. A. Di Pietro, and S. Rebay. An artificial compressibility flux for the discontinuous Galerkin solution of the incompressible Navier-Stokes equations. *J. Comput. Phys.*, 218(2):794–815, 2006.

[4] F. Bassi, S. Rebay, G. Mariotti, S. Pedinotti, and M. Savini. A high-order accurate discontinuous finite element method for inviscid and viscous turbomachinery flows. In R. Decuypere and G. Dibelius, editors, *Proceedings of the* 2nd *European Conference on Turbomachinery Fluid Dynamics and Thermodynamics*, pages 99–109, 1997.

[5] S. C. Brenner. Poincaré-Friedrichs inequalities for piecewise H^1 functions. *SIAM J. Numer. Anal.*, 41(1):306–324, 2003.

[6] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*, volume 15 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2002.

[7] H. Brezis. *Analyse fonctionnelle*. Masson, Paris, 2005 edition, 1983.

[8] F. Brezzi, G. Manzini, D. Marini, P. Pietra, and A. Russo. Discontinuous Galerkin approximations for elliptic problems. *Numer. Methods Partial Differential Equations*, 16(4):365–378, 2000.

[9] B. Cockburn, G. Kanschat, and D. Schötzau. A locally conservative LDG method for the incompressible Navier-Stokes equations. *Math. Comp.*, 74:1067–1095, 2005.

[10] B. Cockburn, G. Kanschat, D. Schötzau, and C. Schwab. Local Discontinuous Galerkin methods for the Stokes system. *SIAM J. Numer. Anal.*, 40(1):319–343 (electronic), 2002.

[11] B. Cockburn and C.-W. Shu. The local discontinuous Galerkin finite element method for convection-diffusion systems. *SIAM J. Numer. Anal.*, 35:2440–2463, 1998.

[12] K. Deimling. *Nonlinear functional analysis*. Springer-Verlag, Berlin, 1985.

[13] D. A. Di Pietro. Analysis of a discontinuous Galerkin approximation of the Stokes problem based on an artificial compressibility flux. *Int. J. Numer. Methods Fluids*, 55:793–813, 2007.

[14] D. A. Di Pietro, A. Ern, and J.-L. Guermond. Discontinuous Galerkin methods for anisotropic semi-definite diffusion with advection. *SIAM J. Numer. Anal.*, 46(2):805–831, 2008.

[15] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, NY, 2004.

[16] A. Ern and J.-L. Guermond. Discontinuous Galerkin methods for Friedrichs' systems. I. General theory. *SIAM J. Numer. Anal.*, 44(2):753–778, 2006.

[17] A. Ern and J.-L. Guermond. Discontinuous Galerkin methods for Friedrichs' systems. II. Second-order elliptic PDEs. *SIAM J. Numer. Anal.*, 44(6):2363–2388, 2006.

[18] A. Ern and J.-L. Guermond. Discontinuous Galerkin methods for Friedrichs' systems. III. Multi-field theories with partial coercivity. *SIAM J. Numer. Anal.*, 46(2):776–804, 2008.

[19] R. Eymard, Th. Gallouët, M. Ghilani, and R. Herbin. Error estimates for the approximate solutions of a nonlinear hyperbolic equation given by finite volume schemes. *IMA J. Numer. Anal.*, 18(4):563–594, 1998.

[20] R. Eymard, Th. Gallouët, and R. Herbin. *The Finite Volume Method*. Ph. Charlet and J.L. Lions eds, North Holland, 2000.

[21] R. Eymard, Th. Gallouët, and R. Herbin. Discretization schemes for heterogeneous and anisotropic diffusion problems on general nonconforming meshes. Available online as HAL report 00203269, January 2008. Submitted for publication.

[22] R. Eymard, Th. Gallouët, R. Herbin, and J.-C. Latché. Analysis tools for finite volume schemes. *Acta Math. Univ. Comenian. (N.S.)*, 76(1):111–136, 2007.

[23] R. Eymard, R. Herbin, and J.-C. Latché. Convergence analysis of a colocated finite volume scheme for the incompressible navier-stokes equations on general 2D or 3D meshes. *SIAM J. Numer. Anal.*, 45(1):1–36, 2007.

[24] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.

[25] V. Girault, B. Rivière, and M. F. Wheeler. A discontinuous Galerkin method with nonoverlapping domain decomposition for the Stokes and Navier-Stokes problems. *Math. Comp.*, 74(249):53–84 (electronic), 2005.

[26] O. A. Karakashian and W. N. Jureidini. A nonconforming finite element method for the stationary Navier-Stokes equations. *SIAM J. Numer. Anal.*, 35(1):93–120 (electronic), 1998.

[27] L. S. G. Kovasznay. Laminar flow behind a two-dimensional grid. *Proc. Camb. Philos. Soc.*, 44:58–62, 1948.

[28] A. Lasis and E. Süli. Poincaré-type inequalities for broken Sobolev spaces. Technical Report 03/10, Oxford University Computing Laboratory, Oxford, England, 2003.

[29] A. Lew, P. Neff, D. Sulsky, and M. Ortiz. Optimal BV estimates for a discontinuous Galerkin method for linear elasticity. *AMRX Appl. Math. Res. Express*, 3:73–106, 2004.

[30] I. Mozolevski, E. Süli, and P. R. Bösing. Discontinuous Galerkin finite element approximation of the two-dimensional Navier-Stokes equations in stream-function formulation. *Comm. Numer. Methods Engrg.*, 23(6):447–459, 2007.

[31] J. Nečas. *Equations aux Dérivées Partielles*. Presses de l'Université de Montréal, Montréal, Canada, 1965.

[32] R. Temam. *Navier-Stokes Equations*, volume 2 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, revised edition, 1979. Theory and numerical analysis, With an appendix by F. Thomasset.

[1]Institut Français du Pétrole, 1 & 4, avenue de Bois-Préau, 92852 Rueil-Malmaison Cedex - France

[2]Université Paris-Est, CERMICS, Ecole des Ponts, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France

E-mail address: daniele-antonio.di-pietro@ifp.fr

E-mail address: ern@cermics.enpc.fr